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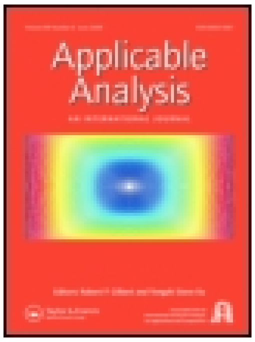

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Lipschitz stability at the boundary for time-harmonic diffuse optical tomography

Olga Doeva[a], Romina Gaburro[b], William R. B. Lionheart[c] and Clifford J. Nolan[b]

[a]Bernal Institute, University of Limerick, Limerick, Ireland; [b]Department of Mathematics and Statistics, Health Research Institute (HRI), University of Limerick, Ireland; [c]Department of Mathematics, University of Manchester, Manchester, UK

ABSTRACT

We study the inverse problem in Optical Tomography of determining the optical properties of a medium $\Omega \subset \mathbb{R}^n$, with $n \geq 3$, under the so-called *diffusion approximation*. We consider the time-harmonic case where Ω is probed with an input field that is modulated with a fixed harmonic frequency $\omega = k/c$, where c is the speed of light and k is the wave number. We prove a result of Lipschitz stability of the *absorption coefficient* μ_a at the boundary $\partial\Omega$ in terms of the measurements in the case when the *scattering coefficient* μ_s is assumed to be known and k belongs to certain intervals depending on some *a-priori* bounds on μ_a, μ_s.

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1. Introduction

Although Maxwell's equations provide a complete model for the light propagation in a scattering medium on a micro scale, on the scale suitable for medical diffuse Optical Tomography (OT) an appropriate model is given by the *radiative transfer equation* (or *Boltzmann equation*) [1]. If Ω is a domain in $\mathbb{R}^n$, with $n \geq 2$ with smooth boundary $\partial\Omega$ and radiation is considered in the body Ω, then it is well known that if the input field is modulated with a fixed harmonic frequency ω, the so-called *diffusion approximation* leads to the complex partial differential equation (see [2]) for the energy current density u

$$-\operatorname{div}(K\nabla u) + (\mu_a - \mathrm{i}k)u = 0, \quad \text{in } \Omega. \tag{1}$$

Here $k = \omega/c$ is the wave number, c is the speed of light and, in the anisotropic case, the so-called *diffusion tensor* K, is the complex matrix-valued function

$$K = \frac{1}{n}\Big((\mu_a - \mathrm{i}k)I + (I - B)\mu_s\Big)^{-1}, \quad \text{in } \Omega, \tag{2}$$

where $B_{ij} = B_{ji}$ is a real matrix-valued function, I is the $n \times n$ identity matrix and $I - B$ is positive definite [2–4] on Ω. The spacially dependent real-valued coefficients μ_a and μ_s are called the *absorption* and the *scattering coefficients* of the medium Ω respectively and represent the optical properties of Ω. It is worth noticing that many tissues including parts of the brain, muscle and breast tissue have

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CONTACT Romina Gaburro romina.gaburro@ul.ie

*This manuscript is dedicated to Sergio Vessella on the occasion of his 65th birthday, to honour his outstanding contribution to the fields of inverse problems and mathematical analysis.

fibrous structure on a microscopic scale which results in anisotropic physical properties on a larger scale. Therefore the model considered in this manuscript seems appropriate for the case of medical applications of OT (see [33]). Although it is common practise in OT to use the Robin-to-Robin map to describe the boundary measurements (see [2]), the Dirichlet-to-Neumann (D–N) map will be employed here instead. This is justified by the fact that in OT, prescribing its inverse, the Neumnann-to-Dirichlet (N-D) map (on the appropriate spaces), is equivalent to prescribing the Robin-to-Robin boundary map. A rigorous definition of the D–N map for Equation (1) will be given in Section 2.

It is also well known that prescribing the N-D map is insufficient to recover both coefficients μ_a and μ_s uniquely [5] unless *a-priori* smoothness assumptions are imposed [6]. In this paper we consider the problem of determining the absorption coefficient μ_a in a medium $\Omega \subset \mathbb{R}^n$, $n \geq 3$, that is probed with an input field which is modulated with a fixed harmonic frequency $\omega = k/c$, with $k \neq 0$ (time-harmonic case) and whose scattering coefficient μ_s is assumed to be known. More precisely, we show that μ_a, restricted to the boundary $\partial\Omega$, depends upon the D–N map of (1), Λ_{K,μ_a}, in a Lipschitz way when k is chosen in certain intervals that depend on *a-priori* bounds on μ_a, μ_s and on the ellipticity constant of $I-B$ (Theorem 2.4). The static case ($k = 0$), for which (1) is a single real elliptic equation, was studied in [7], where the author proved Lipschitz stability of μ_a and Hölder stability of the derivatives of μ_a at the boundary in terms of Λ_{K,μ_a}. In the present paper we show that in the time-harmonic case, for which (1) is a complex elliptic equation, a Lipschitz stability estimate of μ_a at the boundary $\partial\Omega$ in terms of Λ_{K,μ_a} still holds true if k is chosen within certain ranges. The case where μ_a is assumed to be known and the scattering coefficient μ_s is to be determined, can be treated in a similar manner. The choice in this paper of focusing on the determination of μ_a rather than the one of μ_s is driven by the medical application of OT we have in mind. While μ_s varies from tissue to tissue, it is the absorption coefficient μ_a that carries the more interesting physiological information as it is related to the global concentrations of certain metabolites in their oxygenated and deoxygenated states.

Our main result (Theorem 2.4) is based on the construction of singular solutions to the complex elliptic equation (1), having an isolated singularity outside Ω. Such solutions were first constructed in [8] for equations of type

$$\operatorname{div}(K\nabla u) = 0, \quad \text{in } \Omega, \tag{3}$$

when K is a real matrix-valued function belonging to $W^{1,p}(\Omega)$, with $p > n$ and they were employed to prove stability results at the boundary in [8], [9], [10] and [11] in the case of Calderón's problem (see [12]) with global, local data and on manifolds. The singular solutions introduced in [8] were extended in [13] to equations of type

$$-\operatorname{div}(K\nabla u + Pu) + Q \cdot \nabla u + qu = 0, \quad \text{in } \Omega, \tag{4}$$

with real coefficients, where K is merely Hölder continuous. Singular solutions were also studied in [14].

In this paper we extend the singular solutions introduced in [8] to the case of elliptic equations of type (1) with complex coefficients. Such a construction is done by treating (1) as a strongly elliptic system with real coefficients, since $\Re K \geq \tilde{\lambda}^{-1} I > 0$, where $\tilde{\lambda}$ is a positive constant depending on the *a-priori* information on μ_s, B and μ_a. We wish to stress out, however, that in [8] the author constructed singular solutions to (3) which have an isolated singularity of arbitrary high order, where the current paper extends such construction to singular solutions to the complex Equation (1) having an isolated singularity of Green's type only. This is sufficient to prove the Lipschitz continuity of the boundary values of μ_a in terms of the D–N map. The more general construction of the singular solutions with an isolated singularity of arbitrary high order for elliptic complex partial differential equations will be material of future work.

This paper is stimulated by the work of Alessandrini and Vessella [15], where the authors proved global Lipschitz stability of the conductivity in a medium Ω in terms of the D–N map for Calderón's problem, in the case when the conductivity is real, isotropic and piecewise constant on a given partition of Ω. This fundamental result was extended to the complex case in [16] and in the context of various inverse problems for example in [17], [18], [19] and [20], [21], [22] in the isotropic and anisotropic settings, respectively. The machinery of the proof introduced in [15] is based on an induction argument that combines quantitative estimates of unique continuation together with a careful asymptotic analysis of Green's functions. The initial step of their induction argument relies on Lipschitz (or Hölder) stability estimates at the boundary of the physical parameter that one wants to estimate in terms of the boundary measurements, which is the subject of the current manuscript. Our paper also provides a first step towards a reconstruction procedure of μ_a by boundary measurements based on a Landweber iterative method for nonlinear problems studied in [23], where the authors provided an analysis of the convergence of such algorithm in terms of either a Hölder or Lipschitz global stability estimates (see also [24]). We also refer to [25] and [32] for further reconstruction techniques of the optical properties of a medium.

The paper is organised as follows. Section 2 contains the formulation of the problem (Subsections 2.1 and 2.2) and our main result (Subsection 2.3, Theorem 2.4). Section 3 is devoted to the construction of singular solutions of Equation (1) having a Green's type isolated singularity outside Ω. The proof of our main result (Theorem 2.4) is given in Section 4.

2. Formulation of the problem and main result

2.1. Main assumptions

We rigorously formulate the problem by introducing the following notation, definitions and assumptions. For $n \geq 3$, a point $x \in \mathbb{R}^n$ will be denoted by $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. Moreover, given a point $x \in \mathbb{R}^n$, we will denote with $B_r(x), B'_r(x')$ the open balls in $\mathbb{R}^n, \mathbb{R}^{n-1}$, centred at x and x' respectively with radius r and by $Q_r(x)$ the cylinder

$$Q_r(x) = B'_r(x') \times (x_n - r, x_n + r).$$

We will also denote $B_r = B_r(0)$, $B'_r = B'_r(0)$ and $Q_r = Q_r(0)$.

Definition 2.1: Let Ω be a bounded domain in $\mathbb{R}^n$, with $n \geq 3$. We shall say that the boundary of Ω, $\partial\Omega$, is of Lipschitz class with constants $r_0, L > 0$, if for any $P \in \partial\Omega$ there exists a rigid transformation of coordinates under which we have $P = 0$ and

$$\Omega \cap Q_{r_0} = \{(x', x_n) \in Q_{r_0} \mid x_n > \varphi(x')\},$$

where φ is a Lipschitz function on B'_{r_0} satisfying

$$\varphi(0) = 0$$

and

$$\|\varphi\|_{C^{0,1}(B'_{r_0})} \leq Lr_0.$$

We consider, for a fixed $k > 0$,

$$L = -\mathrm{div}\,(K\nabla\cdot) + q, \quad \text{in } \Omega, \tag{5}$$

where K is the complex matrix-valued function

$$K(x) = \frac{1}{n}\Big((\mu_a(x) - \mathrm{i}k)I + (I - B(x))\mu_s(x)\Big)^{-1}, \quad \text{for any } x \in \Omega, \tag{6}$$

and q is the complex-valued function

$$q = \mu_a - ik \quad \text{in } \Omega. \tag{7}$$

We recall that I denotes the $n \times n$ identity matrix, where the matrix B is given by the OT physical experiment and it is such that $B \in L^\infty(\Omega, Sym_n)$, where Sym_n denotes the class of $n \times n$ real-valued symmetric matrices and such that $I-B$ is a positive definite matrix [2–4]. In this paper, we assume that the *scattering coefficient* μ_s is also known in Ω and it is the *absorption coefficient* μ_a that we seek to estimate from boundary measurements.

We assume that there are positive constants λ, E and $\mathcal{E}$ and $p > n$ such that the known quantities B, μ_s and the unknown quantity μ_a satisfy the two assumptions below respectively.

Assumption 2.1 (Assumption on μ_s and B):

$$\lambda^{-1} \leq \mu_s(x) \leq \lambda, \quad \text{for a.e. } x \in \Omega, \tag{8}$$

$$||\mu_s||_{W^{1,p}(\Omega)} \leq E \tag{9}$$

and

$$\mathcal{E}^{-1}|\xi|^2 \leq (I - B(x))\xi \cdot \xi \leq \mathcal{E}|\xi|^2, \quad \text{for a.e. } x \in \Omega, \text{ for any } \xi \in \mathbb{R}^n. \tag{10}$$

Assumption 2.2 (Assumption on μ_a):

$$\lambda^{-1} \leq \mu_a(x), \leq \lambda, \quad \text{for a.e. } x \in \Omega, \tag{11}$$

$$\| \mu_a \|_{W^{1,p}(\Omega)} \leq E. \tag{12}$$

We state below some facts needed in the sequel of the paper. Most of them are straightforward consequences of our assumptions.

The inverse of K

$$K^{-1} = n\Big(\mu_a I + (I - B)\mu_s - ikI\Big), \quad \text{on} \quad \Omega \tag{13}$$

has real and imaginary parts given by the symmetric, real matrix valued-functions on Ω

$$K_R^{-1} = n\left(\mu_a I + (I - B)\mu_s\right), \tag{14}$$

$$K_I^{-1} = -nkI \tag{15}$$

respectively. As an immediate consequence of Assumptions 2.1 and 2.2 we have

$$n\lambda^{-1}(1 + \mathcal{E}^{-1})|\xi|^2 \leq K_R^{-1}(x)\xi \cdot \xi \leq n\lambda(1 + \mathcal{E})|\xi|^2, \tag{16}$$

$$-K_I^{-1}(x)\xi \cdot \xi = nk|\xi|^2, \tag{17}$$

for a.e. $x \in \Omega$ and any $\xi \in \mathbb{R}^n$. Moreover K_R^{-1} and K_I^{-1} commute, therefore the real and imaginary parts of K are the symmetric, real matrix valued-functions on Ω

$$K_R = \frac{1}{n}\left(\Big(\mu_a I + (I - B)\mu_s\Big)^2 + k^2 I\right)^{-1}\Big(\mu_a I + (I - B)\mu_s\Big), \tag{18}$$

$$K_I = \frac{k}{n}\left(\Big(\mu_a I + (I - B)\mu_s\Big)^2 + k^2 I\right)^{-1} \tag{19}$$

respectively. Assumptions 2.1 and 2.2 also imply that

$$K_R(x)\xi\cdot\xi \geq \frac{\lambda(1+\mathcal{E})}{n}\left(\lambda^2(1+\mathcal{E})^2+k^2\right)^{-1}|\xi|^2, \tag{20}$$

$$K_I(x)\xi\cdot\xi \geq \frac{k}{n}\left(\lambda^2(1+\mathcal{E})^2+k^2\right)^{-1}|\xi|^2, \tag{21}$$

for a.e. $x\in\Omega$, for every $\xi\in\mathbb{R}^n$ and the *boundness condition*

$$|K_R(x)|^2+|K_I(x)|^2 \leq \left(\lambda^{-2}(1+\mathcal{E}^{-1})^2+k^2\right)^{-2}\left(\frac{\lambda^2(1+\mathcal{E})^2+k^2}{n^2}\right), \tag{22}$$

for a.e. $x\in\Omega$.

Moreover $K=\{K^{hk}\}_{h,k=1,\dots,n}$ and q satisfy

$$||K^{hk}||_{W^{1,p}(\Omega)} \leq C_1, \quad h,k=1,\dots,n, \tag{23}$$

and

$$|q(x)|=|\mu_a(x)-ik|\leq\lambda+k, \quad \text{for a.e. } x\in\Omega, \tag{24}$$

respectively, where C_1 is a positive constant depending on λ, E, $\mathcal{E}$, k and n.

By denoting $q=q_R+iq_I$, the complex equation

$$-\operatorname{div}(K\nabla u)+qu=0, \quad \text{in } \Omega \tag{25}$$

is equivalent to the system for the vector field $u=(u^1,u^2)$

$$\begin{cases} -\operatorname{div}(K_R\nabla u^1)+\operatorname{div}(K_I\nabla u^2)+\left(q_Ru^1-q_Iu^2\right)=0, & \text{in } \Omega,\\ -\operatorname{div}(K_I\nabla u^1)+\operatorname{div}(K_R\nabla u^2)+\left(q_Iu^1+q_Ru^2\right)=0, & \text{in } \Omega, \end{cases} \tag{26}$$

which can be written in a more compact form as

$$-\operatorname{div}(C\nabla u)+qu=0, \quad \text{in } \Omega \tag{27}$$

or, in components, as

$$-\frac{\partial}{\partial x_h}\left\{C_{lj}^{hk}\frac{\partial}{\partial x_k}u^j\right\}+q_{lj}u^j=0, \qquad \text{for} \quad l=1,2, \quad \text{in } \Omega, \tag{28}$$

where $\{C_{lj}^{hk}\}_{h,k=1,\dots,n}$ is defined by

$$C_{lj}^{hk}=K_R^{hk}\delta_{lj}-K_I^{hk}\left(\delta_{l1}\delta_{j2}-\delta_{l2}\delta_{j1}\right) \tag{29}$$

and $\{q_{lj}\}_{l,j=1,2}$ is a 2×2 real matrix valued function on Ω defined by

$$q_{lj}=q_R\delta_{lj}-q_I\left(\delta_{l1}\delta_{j2}-\delta_{l2}\delta_{j1}\right). \tag{30}$$

(20), together with (22) imply that system (26) is *uniformly elliptic* and *bounded*, therefore it satisfies the *strong ellipticity condition*

$$C_2^{-1}|\xi|^2 \leq C_{lj}^{hk}(x)\xi_h^l\xi_k^j \leq C_2|\xi|^2, \quad \text{for a.e. } x\in\Omega, \quad \text{for all } \xi\in\mathbb{R}^{2n}, \tag{31}$$

where $C_2>0$ is a constant depending on λ, $\mathcal{E}$, k and n.

Remark 2.3: Matrix $q = \{q_{lj}\}_{l,j=1}^{2}$

$$\begin{pmatrix} \mu_a & k \\ -k & \mu_a \end{pmatrix} \tag{32}$$

is uniformly positive definite on Ω and it satisfies

$$\lambda^{-1}|\xi|^2 \leq q(x)\xi \cdot \xi \leq \lambda|\xi|^2, \quad \text{for a.e. } x \in \Omega, \quad \text{for every } \xi \in \mathbb{R}^2. \tag{33}$$

Definition 2.2: We will refer in the sequel to the set of positive numbers r_0, L, λ, E, $\mathcal{E}$ introduced above, along with the space dimension $n, p > n$, the wave number k and the diameter of Ω, $diam(\Omega)$, as to the *a-priori data*.

2.2. The Dirichlet-to-Neumann map

Let K be the complex matrix valued-function on Ω introduced in (6) and $q = \mu_a - ik$, satisfying Assumptions 2.1 and 2.2. B and μ_s are assumed to be known in Ω and satisfying Assumption 2.1, so that K is completely determined by μ_a, satisfying Assumption 2.2, on Ω. Denoting by $\langle\cdot,\cdot\rangle$ the $L^2(\partial\Omega)$-pairing between $H^{1/2}(\partial\Omega)$ and its dual $H^{-1/2}(\partial\Omega)$, we will emphasise such dependence of K on μ_a by denoting K by

$$K_{\mu_a}.$$

For any $v, w \in \mathbb{C}^n$, with $v = (v_1, \ldots, v_n)$, $w = (w_1, \ldots, w_n)$, we will denote throughout this paper by $v \cdot w$, the expression

$$v \cdot w = \sum_{i=1}^{n} v_i w_i.$$

Definition 2.3: The Dirichlet-to-Neumann (D–N) map corresponding to μ_a is the operator

$$\Lambda_{\mu_a} : H^{1/2}(\partial\Omega) \longrightarrow H^{-1/2}(\partial\Omega) \tag{34}$$

defined by

$$\langle \Lambda_{\mu_a} f, \bar{g} \rangle = \int_{\Omega} \Big(K_{\mu_a}(x)\nabla u(x) \cdot \nabla\varphi(x) + (\mu_a(x) - ik)u(x)\varphi(x) \Big)\, \mathrm{d}x, \tag{35}$$

for any $f, g \in H^{1/2}(\partial\Omega)$, where $u \in H^1(\Omega)$ is the weak solution to

$$\begin{cases} -\mathrm{div}(K_{\mu_a}(x)\nabla u(x)) + (\mu_a - ik)(x)u(x) = 0, & \text{in } \Omega, \\ u = f, & \text{on } \partial\Omega \end{cases}$$

and $\varphi \in H^1(\Omega)$ is any function such that $\varphi|_{\partial\Omega} = g$ in the trace sense.

Given B, μ_s, μ_{a_i}, and the corresponding diffusion tensors $K_{\mu_{a_i}}$, for $i = 1, 2$, satisfying Assumptions 2.1 and 2.2, the well known Alessandrini's identity (see [8, (5.0.4), p.129])

$$\langle\Big(\Lambda_{\mu_{a_1}} - \Lambda_{\mu_{a_2}}\Big)f, \overline{g}\rangle = \int_\Omega \Big(K_{\mu_{a_1}}(x) - K_{\mu_{a_2}}(x)\Big)\,\nabla u(x)\cdot\nabla v(x)\,\mathrm{d}x \\ + \int_\Omega \big(\mu_{a_1}(x) - \mu_{a_2}(x)\big)\,u(x)v(x)\,\mathrm{d}x, \tag{36}$$

holds true for any $f, g \in H^{1/2}(\partial\Omega)$, where $u, v \in H^1(\Omega)$ are the unique weak solutions to the Dirichlet problems

$$\begin{cases} -\mathrm{div}(K_{\mu_{a_1}}(x)\nabla u(x)) + (\mu_{a_1} - ik)u(x) = 0, & \text{in } \Omega, \\ u = f, & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} -\mathrm{div}(K_{\mu_{a_2}}(x)\nabla v(x)) + (\mu_{a_2} - ik)v(x) = 0, & \text{in } \Omega, \\ v = g, & \text{on } \partial\Omega, \end{cases}$$

respectively.

We will denote in the sequel by $\|\cdot\|_{\mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))}$ the norm on the Banach space of bounded linear operators between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$.

2.3. The main result

Theorem 2.4 (Lipschitz stability of boundary values): *Let $n \geq 3$, and Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary with constants L, r_0 as in Definition 2.1. If $p > n$, B, μ_s and μ_{a_i}, for $i = 1, 2$, satisfy Assumptions 2.1 and 2.2 and the wave number k satisfies either*

$$0 < k \leq k_0 := \frac{\sqrt{\lambda^2(1+\mathcal{E})^2 + \lambda^{-2}(1+\mathcal{E}^{-1})^2\tan^2\left(\frac{\pi}{2n}\right)} - \lambda(1+\mathcal{E})}{\tan\left(\frac{\pi}{2n}\right)}, \tag{37}$$

or

$$k \geq \tilde{k}_0 := \frac{1 + \sqrt{1 + \tan^2\left(\frac{\pi}{2n}\right)}}{\tan\left(\frac{\pi}{2n}\right)}\lambda(1+\mathcal{E}), \tag{38}$$

where, λ and $\mathcal{E}$ are the positive numbers introduced in Assumptions 2.1 and 2.2, then

$$\| \mu_{a_1} - \mu_{a_2} \|_{L^\infty(\partial\Omega)} \leq C \| \Lambda_{\mu_{a_1}} - \Lambda_{\mu_{a_2}} \|_{\mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))}, \tag{39}$$

where $C > 0$ is a constant depending on n, p, L, r_0, $diam(\Omega)$, λ, E, $\mathcal{E}$ and k.

3. Singular solutions

We consider

$$L = -\mathrm{div}\,(K\nabla\cdot) + q, \quad \text{in } B_R = \Big\{ x \in \mathbb{R}^n \,\Big|\, |x| < R \Big\}, \tag{40}$$

where $K = \{K^{hk}\}_{h,k=1,\dots,n}$ and q are the complex matrix valued-function and the complex function respectively introduced in Section 1 and satisfying Assumptions 2.1 and 2.2 on B_R.

Theorem 3.1 (Singular solutions for $L = -\mathrm{div}(K\nabla\cdot) + q$): *Given L on B_R as in* (40), *there exists* $u \in W^{2,p}_{loc}(B_R \setminus \{0\})$ *such that*

$$Lu = 0, \quad \text{in } B_R \setminus \{0\} \tag{41}$$

and furthermore

$$u(x) = \left(K^{-1}(0)x \cdot x\right)^{2-n/2} + w(x), \tag{42}$$

where w satisfies

$$|w(x)| + |x||Dw(x)| \le C|x|^{2-n+\alpha}, \quad \text{in } B_R \setminus \{0\}, \tag{43}$$

$$\left(\int_{r<|x|<2r} |D^2 w|^p\right)^{1/p} \le Cr^{(n/p)-n+\alpha}, \quad \text{for every } r,\ 0 < r < R/2. \tag{44}$$

Here α is such that $0 < \alpha < 1 - n/p$, and C is a positive constant depending only on $\alpha, n, p, R, \lambda, E, \mathcal{E}$ and k.

Remark 3.2: Since $K^{-1}(0)$ is a complex matrix, the expression

$$\left(K^{-1}(0)x \cdot x\right)^{1/2} \tag{45}$$

appearing in the leading term in (42) is defined as the principal branch of (45), where a branch cut along the negative real axis of the complex plane has been defined for $z^{1/2}, z \in \mathbb{C}$. Expressions like (45) will appear in the sequel of the paper and they will be understood in the same way.

Next we consider two technical lemmas that are needed for the proof of Theorem 3.1. The proofs of these results for the case where $L = -\mathrm{div}(K\nabla\cdot)$, with K a real matrix valued-function, are treated in detail in [8] and their extension to the more general case $L = -\mathrm{div}(K\nabla\cdot) + q$, with K, q a real matrix valued-function and a real function respectively, was extended in [7], therefore only the key points of their proof will be highlighted in the complex case below.

Lemma 3.3: *Let $p > n$ and $u \in W^{2,p}_{loc}(B_R \setminus \{0\})$ be such that, for some positive s,*

$$|u(x)| \le |x|^{2-s}, \quad \text{for any } x \in B_R \setminus \{0\}, \tag{46}$$

$$\left(\int_{r<|x|<2r} |Lu|^p\right)^{1/p} \le Ar^{(n/p)-s}, \quad \text{for any } r,\ 0 < r < \frac{R}{2}. \tag{47}$$

Then we have

$$|Du(x)| \le C|x|^{1-s}, \quad \text{for any } x \in B_R \setminus \{0\}, \tag{48}$$

$$\left(\int_{r<|x|<2r} |D^2 u|^p\right)^{1/p} \le Cr^{(n/p)-s} \quad \text{for any } r,\ 0 < r < \frac{R}{4}, \tag{49}$$

where C is a positive constant depending only on A, n, p, λ, E, $\mathcal{E}$ and k.

Proof of Lemma 3.3: The proof of (49) is based on the interior L^p - Schauder estimate for uniformly elliptic systems

$$\left(\int_{r<|x|<2r} |D^2u|^p\right)^{1/p} \leq C\left\{\left(\int_{(r/2)<|x|<4r} |Lu|^p\right)^{1/p} + r^{-2}\left(\int_{(r/2)<|x|<4r} |u|^p\right)^{1/p}\right\}, \tag{50}$$

for every r, $0 < r < R/4$, which, combined with interpolation inequality

$$r^{(n/p)-1} \sup_{r<|x|<2r} |Du(x)| \leq C\left\{\left(\int_{(r/2)<|x|<4r} |D^2u|^p\right)^{1/p} + r^{-2}\left(\int_{(r/2)<|x|<4r} |u|^p\right)^{1/p}\right. \tag{51}$$

leads to (48). The positive constant C appearing in (50) depends on n, p, λ, E, $\mathcal{E}$ and k only, whereas the positive constant C in (51) depends on n and p only. For (50) we refer to [26, Lemma 6.2.6]) and for a detailed proof of it, in the case of a single real equation in divergence form, we refer to [8, Proof of Lemma 2.1]. We refer to [27, Theorem 5.12] for a detailed proof of (51) in the real case. For the complex case, (51) can be derived by denoting $u = u^1 + iu^2$ and combining

$$\begin{aligned} & r^{(n/p)-1} \sup_{r<|x|<2r} |Du^i(x)| \\ & \leq C\left(||D^2u^i||_{L^p((r/2)<|x|<4r)} + r^{-2}||u^i||_{L^p((r/2)<|x|<4r)}\right) \\ & \leq C\left(||D^2u||_{L^p((r/2)<|x|<4r)} + r^{-2}||u||_{L^p((r/2)<|x|<4r)}\right), \end{aligned} \tag{52}$$

for $i = 1, 2$ together with

$$\sup_{r<|x|<2r} |Du(x)| \leq \sup_{r<|x|<2r} |Du^1(x)| + \sup_{r<|x|<2r} |Du^2(x)|. \tag{53}$$

■

Lemma 3.4: *Let $f \in L^p_{loc}(B_R \setminus \{0\})$ satisfy*

$$\left(\int_{r<|x|<2r} |f|^p\right)^{1/p} \leq Ar^{(n/p)-s}, \quad \text{for any } r,\ 0 < r < \frac{R}{2}, \tag{54}$$

with $2 < s < n < p$. Then there exists $u \in W^{2,p}_{loc}(B_R \setminus \{0\})$ satisfying

$$Lu = f, \quad \text{in } B_R \setminus \{0\} \tag{55}$$

and

$$|u(x)| \leq C|x|^{2-s}, \quad \text{for any } x \in B_R \setminus \{0\}, \tag{56}$$

where C is a positive constant depending only on A, s, n, p, R, λ, E, $\mathcal{E}$ and k.

Proof of Lemma 3.4: If $f \in L^{\infty}(B_R)$ then there exists a unique Green matrix $G(x,y) = \{G_{ij}(x,y)\}_{i,j=1}^{2}$ defined in $\{x, y \in B_R,\ x \neq y\}$ such that

$$LG(\cdot, y) = \delta(\cdot - y)I, \quad \text{for all } y \in B_R \tag{57}$$

in the sense that for every $\phi = (\phi^1, \phi^2) \in C_c^{\infty}(B_R)$ we have

$$\int_{B_R} K_{ij}^{\alpha\beta} D_\beta G_{jk}(\cdot, y) D_\alpha \phi^i + q_{ij} G_{jk}(\cdot, y)\phi^i = \phi^k(y), \quad \text{for } k = 1, 2. \tag{58}$$

Moreover

$$|G(x,y)| \leq C|x-y|^{2-n}, \quad \text{for any } x \neq y, \tag{59}$$

where C is a positive constant depending on n, λ, E, $\mathcal{E}$ and k and the vector valued-function $u = (u^1, u^2)$ defined by

$$u^k(y) = \int_{B_R} G_{jk}(x,y) f^j(x)\, \mathrm{d}x, \quad \text{for } k = 1, 2, \tag{60}$$

satisfies $Lu = f$ with

$$|u(x)| \leq \int_{B_R} |G(x,y)||f(y)|\, \mathrm{d}y \leq C(I_1 + I_2), \tag{61}$$

where $f = (f^1, f^2)$ and

$$I_1 = \int_{|y|<(|x|/2)} |x-y|^{2-n}|f(y)|\, \mathrm{d}y, \tag{62}$$

$$I_2 = \int_{(|x|/2)<|y|<R} |x-y|^{2-n}|f(y)|\, \mathrm{d}y. \tag{63}$$

For the existence, uniqueness and asymptotic behaviour of the Green's matrix G on B_R as in (57)–(59) we refer to [28]. We also refer to [29], [30] and the more recent result [31] for further reading on the issue of the Green's matrix for elliptic systems of the second order. By an argument based on the monotone convergence theorem, one can show that I_1 and I_2 are both bounded from above by $C|x|^{2-s}$, where C is a positive constant depending on A, s, n, p, R, λ, E, $\mathcal{E}$ and k.

If $f^p_{loc}(B_R \setminus \{0\})$, we introduce a sequence $\{f_N\}_{N=1}^{\infty}$, with $f_N = (f_N^1, f_N^2)$, for $N \geq 1$, defined by

$$f_N^j = \begin{cases} N, & \text{when } f^j > N, \\ f^j & \text{when } |f^j| \leq N, \\ -N, & \text{when } f^j < -N, \end{cases}$$

for $j = 1, 2$. $f_N \in L^{\infty}(B_R)$, for any $N \geq 1$ and $f_N \longrightarrow f$ pointwise on $B_R \setminus \{0\}$. For any $N \geq 1$, let $u_N \in W^{2,p}_{loc}(B_R \setminus \{0\})$ be the solution to

$$Lu_N = f_N \quad \text{in } B_R \setminus \{0\} \tag{64}$$

such that

$$|u_N(x)| \leq C_N|x|^{2-s}, \quad \text{for any } x \in B_R \setminus \{0\}. \tag{65}$$

$|f_N| \leq |f|$ on B_R, therefore $||f_N||_{L^p(\tilde{\Omega})} \leq ||f||_{L^p(\tilde{\Omega})}$, for any $\tilde{\Omega}$, $\tilde{\Omega} \subset\subset B_R \setminus \{0\}$, for any $N \geq 1$. By applying interior L^p - Schauder estimates to u_N and using the fact that $f \in L^p_{loc}(B_R \setminus \{0\})$ we obtain that

$$||u_N||_{W^{2,p}(\tilde{\Omega})} \leq C, \quad \text{for any } \tilde{\Omega},\ \tilde{\Omega} \subset\subset B_R \setminus \{0\}, \tag{66}$$

where C is a positive constant that depends on $\tilde{\Omega}$. By applying a diagonal process we can find a subsequence $\{u_N\}_{N=1}^{\infty}$ weakly converging in $W^{2,p}_{loc}(B_R \setminus \{0\})$ to some function $u \in W^{2,p}_{loc}(B_R \setminus \{0\})$. This limit satisfies both (55) and (56). ■

We proceed next with the proof of Theorem 3.1.

Proof of Theorem 3.1.: We start by considering

$$H(x) = C\Big(K^{-1}(0)x \cdot x\Big)^{2-n/2},$$

solution to

$$L_0 H = 0, \quad \text{in } B_R \setminus \{0\}, \tag{67}$$

where $L_0 := -\text{div}(K(0)\nabla \cdot)$ on B_R. We want to find w such that

$$L(H + w) = 0, \quad \text{in } B_R \setminus \{0\}, \tag{68}$$

satisfying (43), (44), where L is defined by (5). We have

$$\begin{aligned} -LH &= -L_0 H - LH \\ &= \Big(K_{ij}(x) - K_{ij}(0)\Big)\frac{\partial^2 H}{\partial x_i \partial x_j} - \frac{\partial a_{ij}}{\partial x_i}\frac{\partial H}{\partial x_j} - qH. \end{aligned} \tag{69}$$

Therefore for any r, $0 < r < R/2$ we have

$$\begin{aligned} \left(\int_{r<|x|<2r} |LH|^p\right)^{1/p} &\leq \left(\int_{r<|x|<2r} |K_{ij}(x) - K_{ij}(0)|^p \left|\frac{\partial^2 H}{\partial x_i \partial x_j}\right|^p\right)^{1/p} \\ &\quad + \left(\int_{r<|x|<2r} \left|\frac{\partial K_{ij}}{\partial x_i}\right|^p \left|\frac{\partial H}{\partial x_j}\right|^p\right)^{1/p} \end{aligned}$$

$$
\begin{aligned}
&+\left(\int_{r<|x|<2r} |qH|^p\right)^{1/p} \\
&\leq \left(\int_{r<|x|<2r} |x|^{\beta p}|x|^{-np}\right)^{1/p} \\
&+\left(\int_{r<|x|<2r} \left|\frac{\partial K_{ij}}{\partial x_i}\right|^p |x|^{(1-n)p}\right)^{1/p} \\
&+\left(\lambda \int_{r<|x|<2r} |x|^{(2-n)p}\right)^{1/p} \\
&\leq Cr^{(n/p)-n+\beta},
\end{aligned} \tag{70}
$$

where $\beta = 1 - n/p$ and C is a positive constant depending on λ, E, $\mathcal{E}$, R and k only. If we take $w \in W^{2,p}_{loc}(B_R \setminus \{0\})$ to be the solution to $Lw = f$ given by Lemma 3.4, with $f = -LH$ and $s = n - \beta$, then

$$|w(x)| \leq C|x|^{2-n+\beta} \tag{71}$$

and, by Lemma 3.3, properties (43), (44) are satisfied. ■

4. Proof of the main result

Since the boundary $\partial\Omega$ is Lipschitz, the normal unit vector field might not be defined on $\partial\Omega$. We shall therefore introduce a unitary vector field $\widetilde{\nu}$ locally defined near $\partial\Omega$ such that: (i) $\widetilde{\nu}$ is C^∞ smooth, (ii) $\widetilde{\nu}$ is non-tangential to $\partial\Omega$ and it points to the exterior of Ω (see [9, Lemmas 3.1–3.3] for a precised construction of $\widetilde{\nu}$). Here we simply recall that any point $z_\tau = x^0 + \tau\widetilde{\nu}$, where $x^0 \in \partial\Omega$, satisfies

$$C\tau \leq d(z_\tau, \partial\Omega) \leq \tau, \quad \text{for any } \tau,\ 0 \leq \tau \leq \tau_0, \tag{72}$$

where τ_0 and C depend on L, r_0 only.

Remark 4.1: Several constants depending on the *a-priori data* introduced in Definition 2.2 will appear in the proof of the main result below. In order to simplify our notation, we shall denote by C any of these constants, avoiding in most cases to point out their specific dependence on the *a-priori data* which may vary from case to case.

Proof of Theorem 2.4.: We start by recalling that by (36) we have

$$
\begin{aligned}
\langle(\Lambda_{\mu_{a_1}} - \Lambda_{\mu_{a_2}})u, \overline{v}\rangle = &\int_\Omega \Big(K_{\mu_{a_1}}(x) - K_{\mu_{a_2}}(x)\Big)\, \nabla u(x) \cdot \nabla v(x)\, \mathrm{d}x \\
&+ \int_\Omega \big(\mu_{a_1}(x) - \mu_{a_2}(x)\big)\, u(x)v(x)\, \mathrm{d}x,
\end{aligned}
$$

for any $u, v \in H^1(\Omega)$ that solve

$$\mathrm{div}\big(K_{\mu_{a_1}}\nabla u\big) + (\mu_{a_1} - ik)u = 0, \quad \text{in } \Omega, \tag{73}$$

$$\mathrm{div}\big(K_{\mu_{a_2}}\nabla v\big) + (\mu_{a_2} - ik)v = 0, \quad \text{in } \Omega. \tag{74}$$

We set $x^0 \in \partial\Omega$ such that

$$(\mu_{a_1} - \mu_{a_2})(x^0) = \| \mu_{a_1} - \mu_{a_2} \|_{L^\infty(\partial\Omega)}$$

and $z_\tau = x^0 + \tau\widetilde{\nu}$, with $0 < \tau \le \tau_0$, where τ_0 is the number fixed in (72). Let $u, v \in W^{2,p}(\Omega)$ be the singular solutions of Theorem 3.1 to (73), (74), respectively, having a singularity at z_τ

$$\begin{aligned} u(x) &= \left(K^{-1}_{\mu_{a_1}}(z_\tau)(x - z_\tau)\cdot(x - z_\tau)\right)^{2-n/2} + O\left(|x - z_\tau|^{2-n+\alpha}\right), \\ v(x) &= \left(K^{-1}_{\mu_{a_2}}(z_\tau)(x - z_\tau)\cdot(x - z_\tau)\right)^{2-n/2} + O\left(|x - z_\tau|^{2-n+\alpha}\right). \end{aligned} \tag{75}$$

By setting $\rho = 2\tau_0$ we have that $B_\rho(z_\tau)\cap\Omega \neq \emptyset$ and from (36) we obtain

$$\begin{aligned} &\| \Lambda_{\mu_{a_1}} - \Lambda_{\mu_{a_2}} \|_{\mathcal{L}(H^{1/2}(\partial\Omega),H^{-1/2}(\partial\Omega))} \|\overline{u}\|_{H^{1/2}(\partial\Omega)}\|v\|_{H^{1/2}(\partial\Omega)} \\ &\quad\ge \left|\int_{\Omega\cap B_\rho(z_\tau)} \left(K_{\mu_{a_1}}(x) - K_{\mu_{a_2}}(x)\right)\nabla u(x)\cdot\nabla v(x)\,\mathrm{d}x\right| \\ &\qquad - \int_{\Omega\setminus B_\rho(z_\tau)} \left|K_{\mu_{a_1}}(x) - K_{\mu_{a_2}}(x)\right| |\nabla u(x)||\nabla v(x)|\,\mathrm{d}x \\ &\qquad - \int_{\Omega\cap B_\rho(z_\tau)} \left|(\mu_{a_1} - \mu_{a_2})(x)\right| |u(x)|\,|v(x)|\,\mathrm{d}x \\ &\qquad - \int_{\Omega\setminus B_\rho(z_\tau)} \left|(\mu_{a_1} - \mu_{a_2})(x)\right| |u(x)|\,|v(x)|\,\mathrm{d}x. \end{aligned} \tag{76}$$

By (75) and Theorem 3.1 we have

$$\begin{aligned} \nabla u(x) &= (2-n)\big(K^{-1}_{\mu_{a_1}}(z_\tau)(x - z_\tau)\cdot(x - z_\tau)\big)^{-n/2} K^{-1}_{\mu_{a_1}}(z_\tau)(x - z_\tau) \\ &\quad + O(|x - z_\tau|^{1-n+\alpha}), \\ \nabla v(x) &= (2-n)\big(K^{-1}_{\mu_{a_2}}(z_\tau)(x - z_\tau)\cdot(x - z_\tau)\big)^{-n/2} K^{-1}_{\mu_{a_2}}(z_\tau)(x - z_\tau) \\ &\quad + O(|x - z_\tau|^{1-n+\alpha}). \end{aligned} \tag{77}$$

Recalling that for $i = 1, 2$ the real and imaginary parts of $K^{-1}_{\mu_{a_i}}$ satisfy (16) and (17), respectively, we have

$$C^{-1}|\xi|^2 \le \left|K^{-1}_{\mu_{a_i}}(x)\xi\cdot\xi\right| \le C|\xi|^2, \quad \text{for a.e. } x \in \Omega, \text{ for every } \xi \in \mathbb{R}^n \tag{78}$$

and combining (76) together with (75), (77) and (78) we obtain

$$\begin{aligned} &\left|\int_{\Omega\cap B_\rho(z_\tau)} \left(K_{\mu_{a_1}}(x) - K_{\mu_{a_2}}(x)\right)\nabla u(x)\cdot\nabla v(x)\,\mathrm{d}x\right| \\ &\quad\le C\left\{\int_{\Omega\cap B_\rho(z_\tau)} |x - z_\tau|^{4-2n}\,\mathrm{d}x + \int_{\Omega\setminus B_\rho(z_\tau)} |x - z_\tau|^{4-2n}\,\mathrm{d}x + \int_{\Omega\setminus B_\rho(z_\tau)} |x - z_\tau|^{2-2n}\,\mathrm{d}x \right. \\ &\qquad \left. + \| \Lambda_{\mu_{a_1}} - \Lambda_{\mu_{a_2}} \|_{\mathcal{L}(H^{1/2}(\partial\Omega),H^{-1/2}(\partial\Omega))} \|u\|_{H^{1/2}(\partial\Omega)}\|v\|_{H^{1/2}(\partial\Omega)}\right\}. \end{aligned} \tag{79}$$

The left-hand side of (79) can be estimated from below by recalling that $K_{\mu_{a_i}}(\cdot)$ is Hölder continuous on $\overline{\Omega}$ with exponent $\beta = 1 - n/p$, for $i = 1, 2$ and by recalling again (75), which leads to

$$\begin{aligned}
&\left|\int_{\Omega\cap B_\rho(z_\tau)} \Big(K_{\mu_{a_1}}(x) - K_{\mu_{a_2}}(x)\Big)\nabla u(x)\cdot\nabla v(x)\,\mathrm{d}x\right| \\
&\quad\geq \left|\int_{\Omega\cap B_\rho(z_\tau)} \Big(K_{\mu_{a_1}}(x^0) - K_{\mu_{a_2}}(x^0)\Big)\nabla u(x)\cdot\nabla v(x)\,\mathrm{d}x\right| \\
&\quad\quad - C\int_{\Omega\cap B_\rho(z_\tau)} |x - x^0|^\beta |\nabla u(x)||\nabla v(x)|\,\mathrm{d}x \\
&\quad\geq \left|\int_{\Omega\cap B_\rho(z_\tau)} \Big(K_{\mu_{a_1}}(x^0) - K_{\mu_{a_2}}(x^0)\Big)\nabla u(x)\cdot\nabla v(x)\,\mathrm{d}x\right| \\
&\quad\quad - C\int_{\Omega\cap B_\rho(z_\tau)} |x - x^0|^\beta |x - z_\tau|^{2-2n}\,\mathrm{d}x
\end{aligned}\tag{80}$$

and combining (80) together with (79) we obtain

$$\begin{aligned}
&\left|\int_{\Omega\cap B_\rho(z_\tau)} \Big(K_{\mu_{a_1}}(x^0) - K_{\mu_{a_2}}(x^0)\Big)\nabla u(x)\cdot\nabla v(x)\,\mathrm{d}x\right| \\
&\quad\leq C\left\{\int_{\Omega\cap B_\rho(z_\tau)} |x - z_\tau|^{2-2n}|x - x^0|^\beta\,\mathrm{d}x + \int_{\Omega\cap B_\rho(z_\tau)} |x - z_\tau|^{4-2n}\,\mathrm{d}x\right. \\
&\quad\quad + \int_{\Omega\setminus B_\rho(z_\tau)} |x - z_\tau|^{4-2n}\,\mathrm{d}x + \int_{\Omega\setminus B_\rho(z_\tau)} |x - z_\tau|^{2-2n}\,\mathrm{d}x \\
&\quad\quad \left. + \parallel \Lambda_{\mu_{a_1}} - \Lambda_{\mu_{a_2}} \parallel_{\mathcal{L}(H^{1/2}(\partial\Omega),H^{-1/2}(\partial\Omega))} \| ||u||_{H^{1/2}(\partial\Omega)}||v||_{H^{1/2}(\partial\Omega)}\right\}.
\end{aligned}\tag{81}$$

Recalling (78) and combining it together with (77), we can estimate the left-hand side of (81) from below as

$$\begin{aligned}
&\left|\int_{\Omega\cap B_\rho(z_\tau)} \Big(K_{\mu_{a_1}}(x^0) - K_{\mu_{a_2}}(x^0)\Big)\nabla u(x)\cdot\nabla v(x)\,\mathrm{d}x\right| \\
&\quad\geq (2-n)^2 \times \left|\int_{\Omega\cap B_\rho(z_\tau)} \frac{K^{-1}_{\mu_{a_2}}(z_\tau)\Big(K_{\mu_{a_1}}(x^0) - K_{\mu_{a_2}}(x^0)\Big)K^{-1}_{\mu_{a_1}}(z_\tau)(x - z_\tau)\cdot(x - z_\tau)}{\big(K^{-1}_{\mu_{a_1}}(z_\tau)(x - z_\tau)\cdot(x - z_\tau)\big)^{n/2}\big(K^{-1}_{\mu_{a_2}}(z_\tau)(x - z_\tau)\cdot(x - z_\tau)\big)^{n/2}}\,\mathrm{d}x\right| \\
&\quad\quad - C\left\{\int_{\Omega\cap B_\rho(z_\tau)} |x - z_\tau|^{2-2n+\alpha} + \int_{\Omega\cap B_\rho(z_\tau)} |x - z_\tau|^{2-2n+2\alpha}\right\}.
\end{aligned}\tag{82}$$

(82) together with (81) leads to

$$\begin{aligned}
&\left|\int_{\Omega\cap B_\rho(z_\tau)} \frac{K^{-1}_{\mu_{a_2}}(z_\tau)\Big(K_{\mu_{a_1}}(x^0) - K_{\mu_{a_2}}(x^0)\Big)K^{-1}_{\mu_{a_1}}(z_\tau)(x - z_\tau)\cdot(x - z_\tau)}{\big(K^{-1}_{\mu_{a_1}}(z_\tau)(x - z_\tau)\cdot(x - z_\tau)\big)^{n/2}\big(K^{-1}_{\mu_{a_2}}(z_\tau)(x - z_\tau)\cdot(x - z_\tau)\big)^{n/2}}\,\mathrm{d}x\right| \\
&\quad\leq C\left\{\int_{\Omega\cap B_\rho(z_\tau)} |x - z_\tau|^{2-2n+\alpha}\,\mathrm{d}x + \int_{\Omega\cap B_\rho(z_\tau)} |x - z_\tau|^{2-2n}|x - x^0|^\beta\,\mathrm{d}x\right.
\end{aligned}$$

$$+\int_{\Omega\cap B_\rho(z_\tau)}|x-z_\tau|^{4-2n}\,\mathrm{d}x+\int_{\Omega\setminus B_\rho(z_\tau)}|x-z_\tau|^{4-2n}\,\mathrm{d}x+\int_{\Omega\setminus B_\rho(z_\tau)}|x-z_\tau|^{2-2n}\,\mathrm{d}x$$
$$\left.+\parallel\Lambda_{\mu_{a_1}}-\Lambda_{\mu_{a_2}}\parallel_{\mathcal{L}(H^{1/2}(\partial\Omega),H^{-1/2}(\partial\Omega))}\|\|u\|_{H^{1/2}(\partial\Omega)}\|v\|_{H^{1/2}(\partial\Omega)}\right\}. \tag{83}$$

$K^{-1}_{\mu_{a_i}}$ is Hölder continuous on $\overline{\Omega}$, with $\beta=1-n/p$, for $i=1,2$ and, recalling that $C\tau\le|x-z_\tau|$, we have

$$\begin{aligned}
&K^{-1}_{\mu_{a_2}}(z_\tau))\Big(K_{\mu_{a_1}}(x^0)-K_{\mu_{a_2}}(x^0)\Big)K^{-1}_{\mu_{a_1}}(z_\tau)(x-z_\tau)\cdot(x-z_\tau)\\
&=\Big(K^{-1}_{\mu_{a_2}}(x^0)+O(\tau^\beta)\Big)\Big(K_{\mu_{a_1}}(x^0)-K_{\mu_{a_2}}(x^0)\Big)\Big(K^{-1}_{\mu_{a_1}}(x^0)+O(\tau^\beta)\Big)(x-z_\tau)\cdot(x-z_\tau)\\
&=\Big(K^{-1}_{\mu_{a_2}}(x^0)-K^{-1}_{\mu_{a_1}}(x^0)\Big)(x-z_\tau)\cdot(x-z_\tau)+O(|x-z_\tau|^{2+\beta})\\
&=n(\mu_{a_2}-\mu_{a_1})(x^0)|x-z_\tau|^2+O(|x-z_\tau|^{2+\beta}).
\end{aligned}\tag{84}$$

Hence (83), combined with (84) and again with (78), leads to

$$\begin{aligned}
&(\mu_{a_1}-\mu_{a_2})(x^0)\\
&\quad\times\left|\int_{\Omega\cap B_\rho(z_\tau)}\frac{|x-z_\tau|^2}{\big(K^{-1}_{\mu_{a_1}}(z_\tau)(x-z_\tau)\cdot(x-z_\tau)\big)^{n/2}\big(K^{-1}_{\mu_{a_2}}(z_\tau)(x-z_\tau)\cdot(x-z_\tau)\big)^{n/2}}\,\mathrm{d}x\right|\\
&\quad\le C\left\{\int_{\Omega\cap B_\rho(z_\tau)}|x-z_\tau|^{2-2n+\beta}\,\mathrm{d}x+\int_{\Omega\cap B_\rho(z_\tau)}|x-z_\tau|^{2-2n+\alpha}\,\mathrm{d}x\right.\\
&\quad+\int_{\Omega\cap B_\rho(z_\tau)}|x-z_\tau|^{2-2n}|x-x^0|^\beta\mathrm{d}x+\int_{\Omega\cap B_\rho(z_\tau)}|x-z_\tau|^{4-2n}\,\mathrm{d}x\\
&\quad+\int_{\Omega\setminus B_\rho(z_\tau)}|x-z_\tau|^{4-2n}\,\mathrm{d}x+\int_{\Omega\setminus B_\rho(z_\tau)}|x-z_\tau|^{2-2n}\,\mathrm{d}x\\
&\quad\left.+\parallel\Lambda_{\mu_{a_1}}-\Lambda_{\mu_{a_2}}\parallel_{\mathcal{L}(H^{1/2}(\partial\Omega),H^{-1/2}(\partial\Omega))}\|\|u\|_{H^{1/2}(\partial\Omega)}\|v\|_{H^{1/2}(\partial\Omega)}\right\}.
\end{aligned}\tag{85}$$

The integrand appearing on the left-hand side of (85) can be expressed as

$$\frac{|x-z_\tau|^2F(x)}{\big|K^{-1}_{\mu_{a_1}}(z_\tau)(x-z_\tau)\cdot(x-z_\tau)\big|^n\big|K^{-1}_{\mu_{a_2}}(z_\tau)(x-z_\tau)\cdot(x-z_\tau)\big|^n},\tag{86}$$

where the complex-valued function F is defined by

$$F(x):=\left\{\Big(\overline{K}^{-1}_{\mu_{a_1}}(z_\tau)(x-z_\tau)\cdot(x-z_\tau)\Big)\Big(\overline{K}^{-1}_{\mu_{a_2}}(z_\tau)(x-z_\tau)\cdot(x-z_\tau)\Big)\right\}^{n/2}.\tag{87}$$

The choices of k in either (37) or (38) imply

$$|\Im F(x)|\le|\Re F(x)|\quad\text{and}\quad\Re F(x)>0,\tag{88}$$

where $\Re z$ and $\Im z$ denote the real and imaginary parts of a complex number z respectively. By combining (88) together with (78), the left-hand side of inequality (85) can be estimated from below as

$$
\begin{aligned}
&(\mu_{a_1}-\mu_{a_2})(x^0)\\
&\quad\times\left|\int_{\Omega\cap B_\rho(z_\tau)}\frac{|x-z_\tau|^2F(x)}{\left|K^{-1}_{\mu_{a_1}}(z_\tau)(x-z_\tau)\cdot(x-z_\tau)\right|^n\left|K^{-1}_{\mu_{a_2}}(z_\tau)(x-z_\tau)\cdot(x-z_\tau)\right|^n}\,\mathrm{d}x\right|\\
&\geq(\mu_{a_1}-\mu_{a_2})(x^0)\\
&\quad\times\Re\left[\int_{\Omega\cap B_\rho(z_\tau)}\frac{|x-z_\tau|^2F(x)}{\left|K^{-1}_{\mu_{a_1}}(z_\tau)(x-z_\tau)\cdot(x-z_\tau)\right|^n\left|K^{-1}_{\mu_{a_2}}(z_\tau)(x-z_\tau)\cdot(x-z_\tau)\right|^n}\,\mathrm{d}x\right]\\
&\geq\frac{1}{\sqrt{2}}(\mu_{a_1}-\mu_{a_2})(x^0)\\
&\quad\times\int_{\Omega\cap B_\rho(z_\tau)}\frac{|x-z_\tau|^2|F(x)|}{\left|K^{-1}_{\mu_{a_1}}(z_\tau)(x-z_\tau)\cdot(x-z_\tau)\right|^n\left|K^{-1}_{\mu_{a_2}}(z_\tau)(x-z_\tau)\cdot(x-z_\tau)\right|^n}\,\mathrm{d}x\\
&\geq\frac{1}{\sqrt{2}}(\mu_{a_1}-\mu_{a_2})(x^0)\\
&\quad\times\int_{\Omega\cap B_\rho(z_\tau)}\frac{|x-z_\tau|^2\left|\overline{K}^{-1}_{\mu_{a_1}}(z_\tau)(x-z_\tau)\cdot(x-z_\tau)\right|^{n/2}\left|\overline{K}^{-1}_{\mu_{a_2}}(z_\tau)(x-z_\tau)\cdot(x-z_\tau)\right|^{n/2}}{\left|K^{-1}_{\mu_{a_1}}(z_\tau)(x-z_\tau)\cdot(x-z_\tau)\right|^n\left|K^{-1}_{\mu_{a_2}}(z_\tau)(x-z_\tau)\cdot(x-z_\tau)\right|^n}\,\mathrm{d}x.
\end{aligned}
\tag{89}
$$

Combing (89) together with (78), we obtain

$$
\begin{aligned}
&(\mu_{a_1}-\mu_{a_2})(x^0)\\
&\quad\times\left|\int_{\Omega\cap B_\rho(z_\tau)}\frac{|x-z_\tau|^2F(x)}{\left|K^{-1}_{\mu_{a_1}}(z_\tau)(x-z_\tau)\cdot(x-z_\tau)\right|^n\left|K^{-1}_{\mu_{a_2}}(z_\tau)(x-z_\tau)\cdot(x-z_\tau)\right|^n}\,\mathrm{d}x\right|\\
&\geq\frac{1}{\sqrt{2}}(\mu_{a_1}-\mu_{a_2})(x^0)C\int_{\Omega\cap B_\rho(z_\tau)}|x-z_\tau|^{2-2n}\,\mathrm{d}x.
\end{aligned}
\tag{90}
$$

(90) combined with (85) and (86) then leads to

$$
\begin{aligned}
&||\mu_{a_1}-\mu_{a_2}||_{L^\infty(\partial\Omega)}\int_{\Omega\cap B_\rho(z_\tau)}|x-z_\tau|^{2-2n}\,\mathrm{d}x.\\
&\quad\leq C\Bigg\{\int_{\Omega\cap B_\rho(z_\tau)}|x-z_\tau|^{2-2n+\beta}\,\mathrm{d}x+\int_{\Omega\cap B_\rho(z_\tau)}|x-z_\tau|^{2-2n+\alpha}\,\mathrm{d}x\\
&\qquad+\int_{\Omega\cap B_\rho(z_\tau)}|x-z_\tau|^{2-2n}|x-x^0|^\beta\,\mathrm{d}x+\int_{\Omega\cap B_\rho(z_\tau)}|x-z_\tau|^{4-2n}\,\mathrm{d}x\\
&\qquad+\int_{\Omega\setminus B_\rho(z_\tau)}|x-z_\tau|^{4-2n}\,\mathrm{d}x+\int_{\Omega\setminus B_\rho(z_\tau)}|x-z_\tau|^{2-2n}\,\mathrm{d}x\\
&\qquad+\parallel\Lambda_{\mu_{a_1}}-\Lambda_{\mu_{a_2}}\parallel_{\mathcal{L}(H^{1/2}(\partial\Omega),H^{-1/2}(\partial\Omega))}\,|||u||_{H^{1/2}(\partial\Omega)}||v||_{H^{1/2}(\partial\Omega)}\Bigg\}.
\end{aligned}
\tag{91}
$$

By recalling (72), the first integral appearing on the right-hand side of (91) can be estimated from above by observing that $\Omega \cap B_\rho(z_\tau) \subset \{x \,|\, C\tau \leq |x - z_\tau| \leq 2\tau_0\}$, therefore

$$\begin{aligned}\int_{\Omega\cap B_\rho(z_\tau)} |x-z_\tau|^{2-2n+\beta}\,\mathrm{d}x &\leq \int_{\{C\tau\leq|x-z_\tau|\leq 2\tau_0\}} |x-z_\tau|^{2-2n+\beta}\,\mathrm{d}x\\ &= \int_{C\tau}^{2\tau_0} s^{2-2n+\beta+n-1}\,\mathrm{d}s \int_{\{|\xi|=1\}} \mathrm{d}S_\xi\\ &\leq C\left((C\tau)^{2-n+\beta} - (2\tau_0)^{2-n+\beta}\right)\\ &\leq C\tau^{2-n+\beta},\end{aligned} \tag{92}$$

(see also [8], [9]), where dS_ξ denotes the surface measure on the unit sphere. Similarly to (92), the second, third and forth integrals on the right-hand side of inequality (91) are estimated from above as

$$\begin{aligned}\int_{\Omega\cap B_\rho(z_\tau)} |x-z_\tau|^{2-2n+\alpha}\,\mathrm{d}x &\leq C\tau^{2-n+\alpha},\\ \int_{\Omega\cap B_\rho(z_\tau)} |x-z_\tau|^{2-2n}|x-x^0|^{\beta}\,\mathrm{d}x &\leq C\tau^{2-n+\beta},\\ \int_{\Omega\cap B_\rho(z_\tau)} |x-z_\tau|^{4-2n}\,\mathrm{d}x &\leq C\tau^{4-n}.\end{aligned} \tag{93}$$

By observing that $(\Omega \setminus B_\rho(z_\tau)) \subset \{x \,|\, 2\tau_0 \leq |x - z_\tau| \leq R\}$, where R depends on $\mathrm{diam}(\Omega)$, the last two integrals appearing on the right-hand side of (91) can be estimated from above as

$$\begin{aligned}\int_{\Omega\setminus B_\rho(z_\tau)} |x-z_\tau|^{4-2n}\,\mathrm{d}x &\leq \int_{\{2\tau_0\leq|x-z_\tau|\leq R\}} |x-z_\tau|^{4-2n}\,\mathrm{d}x \leq C,\\ \int_{\Omega\setminus B_\rho(z_\tau)} |x-z_\tau|^{2-2n}\,\mathrm{d}x &\leq C.\end{aligned} \tag{94}$$

The integral appearing on the left-hand side of (91) can be estimated from below as

$$\int_{\Omega\cap B_\rho(z_\tau)} |x-z_\tau|^{2-2n}\,\mathrm{d}x \geq C\tau^{2-n} \tag{95}$$

and we refer to [13, p.66] for a detailed calculation of estimate (95). By combining (91) together with (92)–(95) and the $H^{1/2}(\partial\Omega)$ norms of u, v (see [8], [9]), we obtain

$$\begin{aligned}&\| \mu_{a_1} - \mu_{a_2} \|_{L^\infty(\partial\Omega)}\, \tau^{2-n}\\ &\quad\leq C\left\{\tau^{2-n+\beta} + \tau^{2-n+\alpha} + \tau^{4-n} + C + \tau^{2-n} \| \Lambda_{\mu_{a_1}} - \Lambda_{\mu_{a_2}} \|_{\mathcal{L}(H^{1/2}(\partial\Omega),H^{-1/2}(\partial\Omega))}\right\}.\end{aligned} \tag{96}$$

By multiplying (96) by τ^{n-2} we obtain

$$\| \mu_{a_1} - \mu_{a_2} \|_{L^\infty(\partial\Omega)} \leq C\left\{\omega(\tau) + \| \Lambda_{\mu_{a_1}} - \Lambda_{\mu_{a_2}} \|_{\mathcal{L}(H^{1/2}(\partial\Omega),H^{-1/2}(\partial\Omega))}\right\}, \tag{97}$$

where $\omega(\tau) \to 0$ as $\tau \to 0$, which concludes the proof. ■

Remark 4.2: When $n = 3$ the ranges for k, (37) and (38), simplify to

$$0 < k \leq k_0 := \sqrt{3\lambda^2(1+\mathcal{E})^2 + \lambda^{-2}(1+\mathcal{E}^{-1})^2} - \sqrt{3}\lambda(1+\mathcal{E}), \tag{98}$$

and

$$k \geq \tilde{k}_0 := (2+\sqrt{3})\lambda(1+\mathcal{E}). \tag{99}$$

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