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A practical formula of solutions for a family of linear non-autonomous fractional nabla difference equations

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Abstract. In this article, we focus on a generalised problem of linear non-autonomous fractional nabla difference equations. Firstly, we define the equations and describe how this family of problems covers other linear fractional difference equations that appear in the literature. Then, by using matrix theory we provide a new practical formula of solutions for these type of equations. Finally, numerical examples are given to justify our theory.

Keywords: non-autonomous, matrix, nabla, fractional, difference equations.

1 Introduction

Difference equations of fractional order have recently proven to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, physics, control, porous media, electromagnetism and so forth, see [5], [6], [15], [20], [24], [25], [30], [32], [33]. At this point it is strongly believed that the fractional discrete operators can have important contribution in generalizing this idea to classical mechanics, non-relativistic quantum mechanics and relativistic quantum field theories.

The theory of discrete fractional equations is also a promising tool for several biological and physical applications where the memory effect appears. The dynamics of the complex systems are better described within this new powerful tool. The nanotechnology and its applications in biology for example as well as the discrete gravity are fields where the fractional discrete models will play an important role in the future, see [6], [15].

There has been a significant development in the study of fractional difference equations and inclusions in recent years; For some recent contributions focusing on the solutions of fractional difference equations, see [1], [2], [3], [4], [7], [8], [9], [10], [11], [14], [16], [17], [18], [21], [23], [27], [29], and the references therein. The stability of fractional difference equations has been studied in [12], [19], [22], [28], [31], [33], [34].

In this article we will use the fractional nabla operator as defined when applied to a sequence. The backward difference operator of first order, denoted by ∇ (nabla operator), when applied to a vector of sequences $Y_k : \mathbb{N} \rightarrow \mathbb{C}^m$ is defined by:

$$\nabla Y_k = Y_k - Y_{k-1};$$

while the backward difference operator of second order, denoted by ∇^2 , is defined by:

$$\nabla^2 Y_k = \nabla(\nabla Y_k) = Y_k - 2Y_{k-1} + Y_{k-2};$$

Similarly, the ν^{th} order backward difference operator, ∇^ν , is defined by:

$$\nabla^\nu Y_k = \frac{1}{\Gamma(\nu+1)} \sum_{j=0}^{\nu} (-1)^j \frac{1}{\Gamma(j+1)\Gamma(\nu-j+1)} Y_{k-j}, \quad \nu \in \mathbb{N}.$$

Where $\Gamma(\cdot)$ is the Gamma function. In order to define the fractional nabla operator, see [2], [3], [4], we set:

$$\nabla^\nu Y_k = f_k.$$

Where f_k , known vector of sequences. By solving for Y_k we get:

$$Y_k = \frac{1}{\Gamma(\nu)} \sum_{j=\alpha}^k (k-j+1)^{\overline{\nu-1}} f_j = \nabla^{-\nu} f_k.$$

Where $b^{\bar{c}} = \frac{\Gamma(b+c)}{\Gamma(b)}$. Based on this expression, i.e. $\nabla^{-\nu} f_k = \frac{1}{\Gamma(\nu)} \sum_{j=\alpha}^k (k-j+1)^{\overline{\nu-1}} f_j$, if we define \mathbb{N}_α by $\mathbb{N}_\alpha = \{\alpha, \alpha+1, \alpha+2, \dots\}$, α positive integer, and n fractional then the nabla fractional operator of n -th order for any $Y_k : \mathbb{N}_\alpha \rightarrow \mathbb{C}^m$ is defined by:

$$\nabla_\alpha^{-n} Y_k = \sum_{j=\alpha}^k b_{k-j} Y_j, \quad (1)$$

where $b_{k-j} = \frac{1}{\Gamma(n)} (k-j+1)^{\overline{n-1}}$, $j = \alpha, \alpha+1, \dots, k-1, k$.

Several type of systems of fractional nabla difference equations have been studied by authors, see [2], [3], [4], [7], [8], [9], [10], [11]. Linear systems can be summarised in the following family of non-autonomous fractional nabla difference equations:

$$F_k \nabla_\alpha^n Y_k = \sum_{j=\alpha}^k G_k^{(j)} Y_j + V_k, \quad k = \alpha+1, \alpha+2, \dots, \quad (2)$$

Where $F_k, G_k^{(j)} : \mathbb{N}_\alpha \rightarrow \mathbb{C}^{r \times m}$, $V_k \in \mathbb{C}^r$ are known matrices and $Y_k : \mathbb{N}_\alpha \rightarrow \mathbb{C}^m$. The symbol (j) on the matrix $G_k^{(j)}$ refers to the fact that this matrix is a coefficient of Y_j in (2). In this article we will focus on the case that $r = m = 1$, i.e. instead of a system we have a generalized non-autonomous difference equation. Note that k in (2) takes values from $\alpha+1$ while Y_k is defined for k greater than α and the matrix equation itself includes Y_α . We will provide more insight on this in the next Section.

Equation (2) covers many linear fractional difference equations that can be found in the literature. For example:

If we replace $F_k = 1, \forall k \in \mathbb{N}_\alpha$; $G_k^{(j)} = 0, \forall j = \alpha, \alpha+1, \dots, k$; $V_k = V$; we get the fractional difference equation

$$\nabla_\alpha^n Y_k = V, \quad \forall k = \alpha+1, \alpha+2, \dots$$

If we replace $F_k = 1, \forall k \in \mathbb{N}_\alpha; G_k^{(k)} = -G, G_k^{(j)} = 0, \forall j \neq k$; we get the fractional difference equation

$$\nabla_\alpha^n Y_k + G Y_k = V_k, \quad \forall k = \alpha + 1, \alpha + 2, \dots$$

If we replace $F_k = F, \forall k \in \mathbb{N}_\alpha; G_k^{(k)} = G, G_k^{(j)} = 0, \forall j \neq k$; we get the fractional difference equation

$$F \nabla_\alpha^n Y_k = G Y_k + V_k, \quad \forall k = \alpha + 1, \alpha + 2, \dots$$

The rest of the paper is organized as follows: in Section 2 we provide our main results, i.e. we obtain a new practical formula of solutions for (2) based on matrix methods and in Section 3 numerical examples are given to justify our theory.

2 Main Results

In this section we will provide our main results. As mentioned in the introduction, in (2) the discrete variable k takes values for $k \geq \alpha + 1$ while Y_k is defined for $k \geq \alpha$. If (2) was holding for $k = \alpha$ then

$$F_\alpha \nabla_\alpha^n Y_\alpha = \sum_{j=\alpha}^{\alpha} G_\alpha^{(j)} Y_j + V_\alpha,$$

or, equivalently, from (1)

$$F_\alpha \sum_{j=\alpha}^{\alpha} b_{\alpha-j} Y_j = \sum_{j=\alpha}^{\alpha} G_\alpha^{(j)} Y_j + V_\alpha,$$

or, equivalently,

$$F_\alpha Y_\alpha = G_\alpha^{(\alpha)} Y_\alpha + V_\alpha,$$

since $b_0 = \frac{1}{\Gamma(-n)} (1)^{-n-1} = \frac{1}{\Gamma(-n)} \frac{\Gamma(1-n-1)}{\Gamma(1)} = 1$. This means that the above equation should hold, which can not always be guaranteed.

We may now present the following Theorem:

Theorem 2.1. Consider (2) for $r = m = 1$ and with a given initial condition Y_α . Then if $F_{\alpha+j} \neq G_{\alpha+j}^{(\alpha+j)}, \forall j \in \mathbb{N}$ there exists always a unique solution for (2) $\forall k = \alpha, \alpha + 1, \dots$ given by:

$$Y_k = (-1)^{k-\alpha} \frac{D_{k-\alpha}^{(0)}}{\prod_{j=1}^{k-\alpha} [F_{\alpha+j} - G_{\alpha+j}^{(\alpha+j)}]} Y_\alpha + \sum_{i=0}^{k-\alpha-1} (-1)^i \frac{D_i^{(k-\alpha-i)} V_{k-i}}{\prod_{j=k-\alpha-i}^{k-\alpha} [F_{\alpha+j} - G_{\alpha+j}^{(\alpha+j)}]}. \quad (3)$$

For $1 \leq i$, $D_i^{(k-\alpha-i)}$ is the following $i \times i$ determinant:

$$D_i^{(k-\alpha-i)} = \begin{vmatrix} A_{k-i+1}^{(k-i)} & A_{k-i+1}^{(k-i+1)} & 0 & 0 & \cdots & 0 \\ A_{k-i+2}^{(k-i)} & A_{k-i+2}^{(k-i+1)} & A_{k-i+2}^{(k-i+2)} & 0 & \cdots & 0 \\ A_{k-i+3}^{(k-i)} & A_{k-i+3}^{(k-i+1)} & A_{k-i+3}^{(k-i+2)} & A_{k-i+3}^{(k-i+3)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{k-1}^{(k-i)} & A_{k-1}^{(k-i+1)} & A_{k-1}^{(k-i+2)} & A_{k-1}^{(k-i+3)} & \cdots & A_{k-1}^{(k-1)} \\ A_k^{(k-i)} & A_k^{(k-i+1)} & A_k^{(k-i+2)} & A_k^{(k-i+3)} & \cdots & A_k^{(k-1)} \end{vmatrix} \quad (4)$$

and $D_0^{(k-\alpha-i)} = 1$. Where

$$A_k^{(j)} = \frac{1}{\Gamma(-n)} (k-j+1)^{-n-1} (F_k - G_k^{(j)}), \quad \forall j = \alpha, \alpha+1, \dots, k \quad \text{and} \quad k = \alpha+1, \alpha+2, \dots \quad (5)$$

Proof. Since $r = m = 1$ we have a fractional nabla difference equation. By replacing (1) into (2) we get:

$$F_k \left[\sum_{j=\alpha}^k \frac{1}{\Gamma(-n)} (k-j+1)^{-n-1} Y_j \right] = \sum_{j=\alpha}^k G_k^{(j)} Y_j + V_k, \quad k = \alpha+1, \alpha+2, \dots,$$

or, equivalently,

$$\sum_{j=\alpha}^k \left(\frac{1}{\Gamma(-n)} (k-j+1)^{-n-1} F_k - G_k^{(j)} \right) Y_j = V_k, \quad k = \alpha+1, \alpha+2, \dots$$

If we set

$$A_k^{(j)} = \frac{1}{\Gamma(-n)} (k-j+1)^{-n-1} F_k - G_k^{(j)}, \quad \forall j = \alpha, \alpha+1, \dots, k \quad \text{and} \quad k = \alpha+1, \alpha+2, \dots,$$

the previous expression will take the form:

$$\sum_{j=\alpha}^k A_k^{(j)} Y_j = V_k, \quad k = \alpha+1, \alpha+2, \dots \quad (6)$$

We can define any two $A_\alpha^{(\alpha)}, V_\alpha \in \mathbb{C}$ such that:

$$Y_\alpha = \frac{V_\alpha}{A_\alpha^{(\alpha)}}. \quad (7)$$

Hence we have:

$$A_\alpha^{(\alpha)} Y_\alpha = V_\alpha;$$

By replacing k with $k = \alpha+1, \alpha+2, \dots, \alpha+M$, $M \in \mathbb{N}$ random, in (6) we get:

$$A_{\alpha+1}^{(\alpha)} Y_\alpha + A_{\alpha+1}^{(\alpha+1)} Y_{\alpha+1} = V_{\alpha+1}, \quad \text{for} \quad k = \alpha+1;$$

$$A_{\alpha+2}^{(\alpha)} Y_\alpha + A_{\alpha+2}^{(\alpha+1)} Y_{\alpha+1} + A_{\alpha+2}^{(\alpha+2)} Y_{\alpha+2} = V_{\alpha+2}, \quad \text{for} \quad k = \alpha+2;$$

\vdots

$$A_N^{(\alpha)} Y_\alpha + A_N^{(\alpha+1)} Y_{\alpha+1} + \dots + A_N^{(N)} Y_N = V_N, \quad \text{for} \quad k = \alpha+M = N.$$

Equivalently, the above equations can be written in matrix form as:

$$\begin{bmatrix} A_{\alpha}^{(\alpha)} & 0 & 0 & \cdots & 0 \\ A_{\alpha+1}^{(\alpha)} & A_{\alpha+1}^{(\alpha+1)} & 0 & \cdots & 0 \\ A_{\alpha+2}^{(\alpha)} & A_{\alpha+2}^{(\alpha+1)} & A_{\alpha+2}^{(\alpha+2)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_N^{(\alpha)} & A_N^{(\alpha+1)} & A_N^{(\alpha+2)} & \cdots & A_N^{(N)} \end{bmatrix} \begin{bmatrix} Y_{\alpha} \\ Y_{\alpha+1} \\ Y_{\alpha+2} \\ \vdots \\ Y_N \end{bmatrix} = \begin{bmatrix} V_{\alpha} \\ V_{\alpha+1} \\ V_{\alpha+2} \\ \vdots \\ V_N \end{bmatrix}.$$

The above expression is an algebraic system of $M + 1$ equations, $M + 1$ unknowns (where $N = \alpha + M$) and has a unique solution if and only if the determinant of the matrix is non-zero, i.e.:

$$A_k^{(k)} \neq 0, \quad \forall k = \alpha, \alpha + 1, \dots$$

Then the matrix is invertible (its a lower triangular matrix) and the solution of the system by inverting this matrix is given by:

$$\begin{bmatrix} Y_{\alpha} \\ Y_{\alpha+1} \\ Y_{\alpha+2} \\ \vdots \\ Y_N \end{bmatrix} = \begin{bmatrix} \frac{V_{\alpha}}{A_{\alpha}^{(\alpha)}} \\ -\frac{A_{\alpha+1}^{(\alpha)}}{A_{\alpha}^{(\alpha)} A_{\alpha+1}^{(\alpha+1)}} V_{\alpha} + \frac{1}{A_{\alpha+1}^{(\alpha+1)}} V_{\alpha+1} \\ \frac{A_{\alpha+1}^{(\alpha)} A_{\alpha+2}^{(\alpha+1)} - A_{\alpha+1}^{(\alpha+1)} A_{\alpha+2}^{(\alpha)}}{A_{\alpha}^{(\alpha)} A_{\alpha+1}^{(\alpha+1)} A_{\alpha+2}^{(\alpha+2)}} V_{\alpha} - \frac{A_{\alpha+2}^{(\alpha+1)}}{A_{\alpha+1}^{(\alpha+1)} A_{\alpha+2}^{(\alpha+2)}} V_{\alpha+1} + \frac{1}{A_{\alpha+2}^{(\alpha+2)}} V_{\alpha+2} \\ \vdots \\ \sum_{i=0}^M (-1)^i \frac{D_i^{(M-i)} V_{M+\alpha-i}}{\prod_{j=M-i}^M A_{\alpha+j}^{(\alpha+j)}} \end{bmatrix}.$$

Where

$$Y_N = Y_{\alpha+M} = \sum_{i=0}^M (-1)^i \frac{D_i^{(M-i)} V_{M+\alpha-i}}{\prod_{j=M-i}^M A_{\alpha+j}^{(\alpha+j)}}.$$

For the determinant $D_i^{(M-i)}$ we have that $D_0^{(M)} = 1$ and $\forall i \geq 1$, see [13], [26] its values are given by:

$$D_i^{(M-i)} = \begin{vmatrix} A_{\alpha+M-i+1}^{(\alpha+M-i)} & A_{\alpha+M-i+1}^{(\alpha+M-i+1)} & 0 & 0 & \cdots & 0 \\ A_{\alpha+M-i+2}^{(\alpha+M-i)} & A_{\alpha+M-i+2}^{(\alpha+M-i+1)} & A_{\alpha+M-i+2}^{(\alpha+M-i+2)} & 0 & \cdots & 0 \\ A_{\alpha+M-i+3}^{(\alpha+M-i)} & A_{\alpha+M-i+3}^{(\alpha+M-i+1)} & A_{\alpha+M-i+3}^{(\alpha+M-i+2)} & A_{\alpha+M-i+3}^{(\alpha+M-i+3)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ A_{\alpha+M-1}^{(\alpha+M-i)} & A_{\alpha+M-1}^{(\alpha+M-i+1)} & A_{\alpha+M-1}^{(\alpha+M-i+2)} & A_{\alpha+M-1}^{(\alpha+M-i+3)} & \cdots & A_{\alpha+M-1}^{(\alpha+M-1)} \\ A_{\alpha+M}^{(\alpha+M-i)} & A_{\alpha+M}^{(\alpha+M-i+1)} & A_{\alpha+M}^{(\alpha+M-i+2)} & A_{\alpha+M}^{(\alpha+M-i+3)} & \cdots & A_{\alpha+M}^{(\alpha+M-1)} \end{vmatrix}.$$

By generalizing the above results we have:

$$Y_k = \sum_{i=0}^{k-\alpha} (-1)^i \frac{D_i^{(k-\alpha-i)} V_{k-i}}{\prod_{j=k-\alpha-i}^{k-\alpha} A_{\alpha+j}^{(\alpha+j)}}, \quad \forall k = \alpha, \alpha + 1, \dots$$

Where $D_0^{(k)} = 1$ and for $1 \leq i$, $D_i^{(k-i)}$ is a $i \times i$ determinant given by (4). The elements $A_k^{(j)}$, $\forall j = \alpha, \alpha + 1, \dots, k$ and $k = \alpha + 1, \alpha + 2, \dots$ are given by (3). The above sum can be written as:

$$Y_k = \sum_{i=0}^{k-\alpha-1} (-1)^i \frac{D_i^{(k-\alpha-i)} V_{k-i}}{\prod_{j=k-\alpha-i}^{k-\alpha} A_{\alpha+j}^{(\alpha+j)}} + \frac{(-1)^{k-\alpha} D_{k-\alpha}^{(k-\alpha-(k-\alpha))} V_{k-(k-\alpha)}}{\prod_{j=k-\alpha-(k-\alpha)}^{k-\alpha} A_{\alpha+j}^{(\alpha+j)}},$$

or, equivalently,

$$Y_k = (-1)^{k-\alpha} \frac{D_{k-\alpha}^{(0)} V_{\alpha}}{\prod_{j=0}^{k-\alpha} A_{\alpha+j}^{(\alpha+j)}} + \sum_{i=0}^{k-\alpha-1} (-1)^i \frac{D_i^{(k-\alpha-i)} V_{k-i}}{\prod_{j=k-\alpha-i}^{k-\alpha} A_{\alpha+j}^{(\alpha+j)}},$$

or, equivalently,

$$Y_k = (-1)^{k-\alpha} \frac{D_{k-\alpha}^{(0)} V_{\alpha}}{\left[\prod_{j=1}^{k-\alpha} A_{\alpha+j}^{(\alpha+j)} \right] A_{\alpha}^{(\alpha)}} + \sum_{i=0}^{k-\alpha-1} (-1)^i \frac{D_i^{(k-\alpha-i)} V_{k-i}}{\prod_{j=k-\alpha-i}^{k-\alpha} A_{\alpha+j}^{(\alpha+j)}}.$$

And since we set (7), by substituting it in the above expression we get

$$Y_k = (-1)^{k-\alpha} \frac{D_{k-\alpha}^{(0)}}{\prod_{j=1}^{k-\alpha} A_{\alpha+j}^{(\alpha+j)}} Y_{\alpha} + \sum_{i=0}^{k-\alpha-1} (-1)^i \frac{D_i^{(k-\alpha-i)} V_{k-i}}{\prod_{j=k-\alpha-i}^{k-\alpha} A_{\alpha+j}^{(\alpha+j)}},$$

From (5) for $j = k$ we have

$$A_k^{(k)} = \frac{1}{\Gamma(-n)} (1)^{-n-1} \left[F_k - G_k^{(k)} \right],$$

or, equivalently,

$$A_k^{(k)} = F_k - G_k^{(k)}, \quad \forall k = \alpha + 1, \alpha + 2, \dots$$

Hence

$$A_{\alpha+j}^{(\alpha+j)} = F_{\alpha+j} - G_{\alpha+j}^{(\alpha+j)}.$$

And thus $\forall k = \alpha, \alpha + 1, \dots$ we have

$$Y_k = (-1)^{k-\alpha} \frac{D_{k-\alpha}^{(0)}}{\prod_{j=1}^{k-\alpha} [F_{\alpha+j} - G_{\alpha+j}^{(\alpha+j)}]} Y_\alpha + \sum_{i=0}^{k-\alpha-1} (-1)^i \frac{D_i^{(k-\alpha-i)} V_{k-i}}{\prod_{j=k-\alpha-i}^{k-\alpha} [F_{\alpha+j} - G_{\alpha+j}^{(\alpha+j)}]}.$$

The proof is completed.

Remark 2.1. We consider (2) for $r = m = 1$ but with the initial condition Y_α not given. Then we can set $Y_\alpha = C$, $C \in \mathbb{C}$ constant and if $F_{\alpha+j} \neq G_{\alpha+j}^{(\alpha+j)}$, $\forall j \in \mathbb{N}$ there exist always solutions for (2) $\forall k = \alpha, \alpha + 1, \dots$ given by:

$$Y_k = (-1)^{k-\alpha} \frac{D_{k-\alpha}^{(0)}}{\prod_{j=1}^{k-\alpha} [F_{\alpha+j} - G_{\alpha+j}^{(\alpha+j)}]} C + \sum_{i=0}^{k-\alpha-1} (-1)^i \frac{D_i^{(k-\alpha-i)} V_{k-i}}{\prod_{j=k-\alpha-i}^{k-\alpha} [F_{\alpha+j} - G_{\alpha+j}^{(\alpha+j)}]}.$$

For $1 \leq i$, $D_i^{(k-\alpha-i)}$ is given by (4), $D_0^{(k-\alpha-i)} = 1$ and $A_k^{(j)}$, $\forall j = \alpha, \alpha + 1, \dots, k$, $k = \alpha + 1, \alpha + 2, \dots$ is given by (5).

Corollary 2.1. We consider (2) for $r = m = 1$, with given initial condition Y_α and $V_k = 0$, i.e.

$$F_k \nabla_\alpha^n Y_k = \sum_{j=\alpha}^k G_k^{(j)} Y_j, \quad k = \alpha + 1, \alpha + 2, \dots, \quad (8)$$

Then (8) is the homogeneous fractional nabla difference equation of (2) and if $F_{\alpha+j} \neq G_{\alpha+j}^{(\alpha+j)}$, $\forall j \in \mathbb{N}$ there exists always a unique solution $\forall k = \alpha + 1, \alpha + 2, \dots$ given by:

$$Y_k = (-1)^{k-\alpha} \frac{D_{k-\alpha}^{(0)}}{\prod_{j=1}^{k-\alpha} [F_{\alpha+j} - G_{\alpha+j}^{(\alpha+j)}]} Y_\alpha \quad (9)$$

For $1 \leq i$, $D_i^{(k-\alpha-i)}$ is given by (4), $D_0^{(k-\alpha-i)} = 1$ and $A_k^{(j)}$, $\forall j = \alpha, \alpha + 1, \dots, k$, $k = \alpha + 1, \alpha + 2, \dots$ is given by (5).

Remark 2.2. The determinant (4) is easy to construct and understand. Suppose we want to compute the determinant $D_q^{(p)}$ at $k = \alpha + M$ for random $q, p \in \mathbb{N}$. For $q = 0$ we have $D_0^{(p)} = 1$. For $q \geq 1$, we know that we have a

$q \times q$ determinant which is the following:

$$D_m^{(p)} = \begin{vmatrix} A_{\alpha+M-q+1}^{(\alpha+p)} & A_{\alpha+M-q+1}^{(\alpha+p+1)} & 0 & 0 & \cdots & 0 \\ A_{\alpha+M-q+2}^{(\alpha+p)} & A_{\alpha+M-q+2}^{(\alpha+p+1)} & A_{\alpha+M-q+2}^{(\alpha+p+2)} & 0 & \cdots & 0 \\ A_{\alpha+M-q+3}^{(\alpha+p)} & A_{\alpha+M-q+3}^{(\alpha+p+1)} & A_{\alpha+M-q+3}^{(\alpha+p+2)} & A_{\alpha+M-q+3}^{(\alpha+p+3)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{\alpha+M-1}^{(\alpha+p)} & A_{\alpha+M-1}^{(\alpha+p+1)} & A_{\alpha+M-1}^{(\alpha+p+2)} & A_{\alpha+M-1}^{(\alpha+p+3)} & \cdots & A_{\alpha+M-1}^{(\alpha+p+q-1)} \\ A_{\alpha+M}^{(\alpha+p)} & A_{\alpha+M}^{(\alpha+p+1)} & A_{\alpha+M}^{(\alpha+p+2)} & A_{\alpha+M}^{(\alpha+p+3)} & \cdots & A_{\alpha+M}^{(\alpha+p+q-1)} \end{vmatrix}.$$

If d_{ij} is an element of the determinant then $d_{ij} = 0$ for $j - i \geq 2$.

Remark 2.3. In Theorem 2.1 we require that $F_{\alpha+j} \neq G_{\alpha+j}^{(\alpha+j)}$, $\forall j \in \mathbb{N}$. This is because of a lower triangular matrix that is used in the proof and has to be regular in order to use its inverse. However, in the rare case that this doesn't hold and the matrix that appears in the Theorem is singular, we can use optimization methods to have its pseudoinverse.

Now we will consider the system (2) but for $k \in \mathbb{N}_{\alpha+2}$, i.e. the discrete variable k in the system takes values for $k \geq \alpha + 2$ while Y_k is defined for $k \geq \alpha$.

Corollary 2.2. Consider (2) with $r = m = 1$, for $k = \alpha + 2, \alpha + 3, \dots$ and with given initial conditions $Y_\alpha, Y_{\alpha+1}$. Then if $F_{\alpha+j} \neq G_{\alpha+j}^{(\alpha+j)}$, $\forall j \in \mathbb{N}$ there exists always a unique solution for (2) $\forall k = \alpha + 2, \alpha + 3, \dots$ given by:

$$Y_k = (-1)^{k-\alpha} \frac{D_{k-\alpha}^{(0)} Y_\alpha - D_{k-\alpha-1}^{(1)} Y_{\alpha+1}}{\prod_{j=2}^{k-\alpha} [F_{\alpha+j} - G_{\alpha+j}^{(\alpha+j)}]} + \sum_{i=0}^{k-\alpha-2} \frac{(-1)^i D_i^{(k-\alpha-i)} V_{k-i}}{\prod_{j=k-\alpha-i}^{k-\alpha} [F_{\alpha+j} - G_{\alpha+j}^{(\alpha+j)}]} \quad (10)$$

For $1 \leq i$, $D_i^{(k-\alpha-i)}$ is the following $i \times i$ determinant given in (4) and $D_0^{(k-\alpha-i)} = 1$. Where $A_\alpha^{(\alpha)} = A_{\alpha+1}^{(\alpha+1)} = 1$, $A_{\alpha+1}^{(\alpha)} = 0$ and $A_k^{(j)}$, $\forall j = \alpha, \alpha + 1, \dots, k$, $k = \alpha + 2, \alpha + 3, \dots$ is given by (5).

Proof. By replacing (1) into (2) and setting (5) $\forall j = \alpha, \alpha + 1, \dots, k$ and $k = \alpha + 2, \alpha + 3, \dots$, we arrive after similar computations as in the proof of Theorem 2.1 at (6). We can define the random values $A_\alpha^{(\alpha)}, V_\alpha, A_{\alpha+1}^{(\alpha+1)}, V_{\alpha+1}, A_{\alpha+1}^{(\alpha)} \in \mathbb{C}$ such that:

$$Y_\alpha = V_\alpha, \quad Y_{\alpha+1} = V_{\alpha+1}, \quad A_\alpha^{(\alpha)} = A_{\alpha+1}^{(\alpha+1)} = 1, \quad A_{\alpha+1}^{(\alpha)} = 0. \quad (11)$$

Hence we have:

$$Y_\alpha = V_\alpha \quad \text{and} \quad Y_{\alpha+1} = V_{\alpha+1}.$$

By replacing k with $k = \alpha + 2, \alpha + 3, \dots, \alpha + M$, $M \in \mathbb{N}$ random, in (6) we get:

$$\begin{aligned} A_{\alpha+2}^{(\alpha)} Y_\alpha + A_{\alpha+2}^{(\alpha+1)} Y_{\alpha+1} + A_{\alpha+2}^{(\alpha+2)} Y_{\alpha+2} &= V_{\alpha+2}, \quad \text{for } k = \alpha + 2; \\ A_{\alpha+3}^{(\alpha)} Y_\alpha + A_{\alpha+3}^{(\alpha+1)} Y_{\alpha+1} + A_{\alpha+3}^{(\alpha+2)} Y_{\alpha+2} + A_{\alpha+3}^{(\alpha+3)} Y_{\alpha+3} &= V_{\alpha+3}, \quad \text{for } k = \alpha + 3; \\ &\vdots \\ A_N^{(\alpha)} Y_\alpha + A_N^{(\alpha+1)} Y_{\alpha+1} + \dots + A_N^{(N)} Y_N &= V_N, \quad \text{for } k = \alpha + M = N. \end{aligned}$$

Equivalently, the above equations can be written in matrix form as:

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ A_{\alpha+2}^{(\alpha)} & A_{\alpha+2}^{(\alpha+1)} & A_{\alpha+2}^{(\alpha+2)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_N^{(\alpha)} & A_N^{(\alpha+1)} & A_N^{(\alpha+2)} & \cdots & A_N^{(N)} \end{bmatrix} \begin{bmatrix} Y_\alpha \\ Y_{\alpha+1} \\ Y_{\alpha+2} \\ \vdots \\ Y_N \end{bmatrix} = \begin{bmatrix} V_\alpha \\ V_{\alpha+1} \\ V_{\alpha+2} \\ \vdots \\ V_N \end{bmatrix}.$$

The above expression is an algebraic system of $M + 1$ equations, $M + 1$ unknowns (where $N = \alpha + M$) and has a unique solution if and only if the determinant of the matrix is non-zero, i.e. $A_k^{(k)} \neq 0$, $\forall k = \alpha + 2, \alpha + 3, \dots$. Then the matrix is invertible (its a lower triangular matrix) and from the solution of the system by inverting this matrix and working in a similar way as in the the proof of Theorem 2.1 we arrive at the solution:

$$Y_k = \sum_{i=0}^{k-\alpha} (-1)^i \frac{D_i^{(k-\alpha-i)} V_{k-i}}{\prod_{j=k-\alpha-i}^{k-\alpha} A_{\alpha+j}^{(\alpha+j)}}, \quad \forall k = \alpha, \alpha + 1, \dots$$

Note that from (11), $A_\alpha^{(\alpha)} = A_{\alpha+1}^{(\alpha+1)} = 1$ and $A_{\alpha+1}^{(\alpha)} = 0$. Also, where $D_0^{(k)} = 1$ and for $1 \leq i$, $D_i^{(k-i)}$ is a $i \times i$ determinant given by (4). The elements $A_k^{(j)}$, $\forall j = \alpha, \alpha + 1, \dots, k$ and $k = \alpha + 2, \alpha + 3, \dots$ are given by (3). The above sum can be written as:

$$Y_k = \sum_{i=0}^{k-\alpha-2} \frac{(-1)^i D_i^{(k-\alpha-i)} V_{k-i}}{\prod_{j=k-\alpha-i}^{k-\alpha} A_{\alpha+j}^{(\alpha+j)}} - \frac{(-1)^{k-\alpha} D_{k-\alpha-1}^{(1)} V_{\alpha+1}}{\prod_{j=1}^{k-\alpha} A_{\alpha+j}^{(\alpha+j)}} + \frac{(-1)^{k-\alpha} D_{k-\alpha}^{(0)} V_\alpha}{\prod_{j=0}^{k-\alpha} A_{\alpha+j}^{(\alpha+j)}},$$

or, equivalently,

$$Y_k = \frac{(-1)^{k-\alpha} D_{k-\alpha}^{(0)}}{\prod_{j=2}^{k-\alpha} A_{\alpha+j}} Y_\alpha - \frac{(-1)^{k-\alpha} D_{k-\alpha-1}^{(1)}}{\prod_{j=2}^{k-\alpha} A_{\alpha+j}} Y_{\alpha+1} + \sum_{i=0}^{k-\alpha-2} \frac{(-1)^i D_i^{(k-\alpha-i)} V_{k-i}}{\prod_{j=k-\alpha-i}^{k-\alpha} A_{\alpha+j}}.$$

From the above expression we conclude to (10). The proof is completed.

3 Numerical Examples

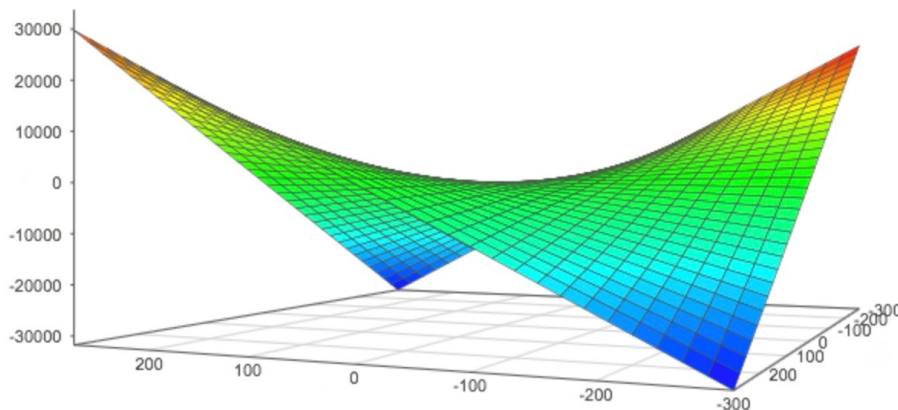


Figure 1: A 3D graph for Y_3 when G (bottom left axis) and the initial conditions (bottom right axis) take values in the interval $[-300, 300]$.

In this Section we will provide numerical examples to justify our theory. As it will be seen, the formulas (3), (9) and (10) are very easy to use since they do not consist of infinite series as in other formulas that appear in the literature.

Example 3.1.

We assume the following fractional difference equation for $k = 1, 2, \dots$

$$\nabla_0^{\frac{1}{2}} Y_k = G Y_k + k^2$$

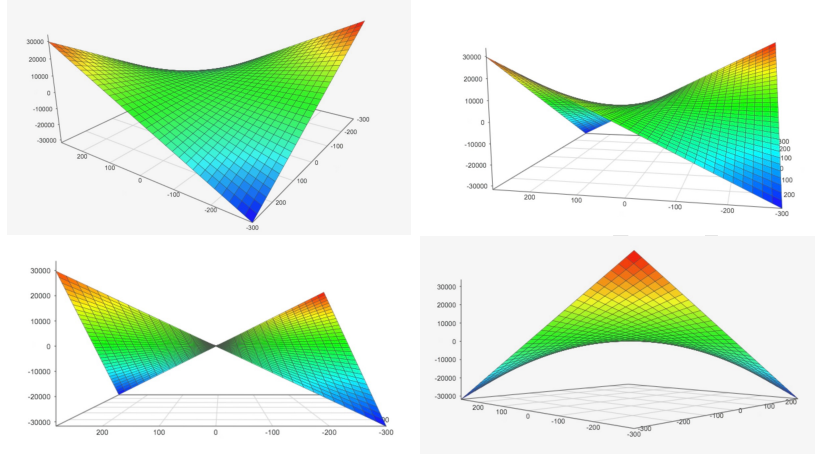


Figure 2: Graphs for Y_3 from different perspective of view, when G and the initial conditions take values in the interval $[-300, 300]$.

with $Y_k : \mathbb{N} \rightarrow \mathbb{C}$ and $G \in \mathbb{C}$ with $G \neq 1$. Then the above equation belongs in the family of (2) for:

$$\begin{aligned} \alpha &= 0; \\ F_k &= 1, \quad \forall k \in \mathbb{N}; \\ G_k^{(k)} &= G \quad \text{and} \quad G_k^{(j)} = 0, \quad \forall j \neq k; \\ A_k^{(k)} &= 1 - G \quad \text{and} \quad A_k^{(j)} = \frac{1}{\Gamma(-0.5)}(k - j + 1)^{-1.5} = \frac{\Gamma(k - j - 0.5)}{\Gamma(-0.5)\Gamma(k - j)}, \quad \forall j \neq k; \\ V_k &= k^2. \end{aligned}$$

Since $G \neq 1$ from Theorem 2.1 there exists a unique solution given by (3):

$$Y_k = (-1)^k \frac{D_k^{(0)}}{1 - G} Y_0 + \sum_{i=0}^{k-1} (-1)^i \frac{D_i^{(k-i)}(k-i)^2}{\prod_{j=k-i}^k [1 - G]}.$$

The above formula is very practical and easy to use. For example at $k = 3$ we get:

$$Y_3 = -\frac{D_3^{(0)}}{1 - G} Y_0 + \sum_{i=0}^2 (-1)^i \frac{D_i^{(3-i)}(3-i)^2}{\prod_{j=3-i}^3 [1 - G]}.$$

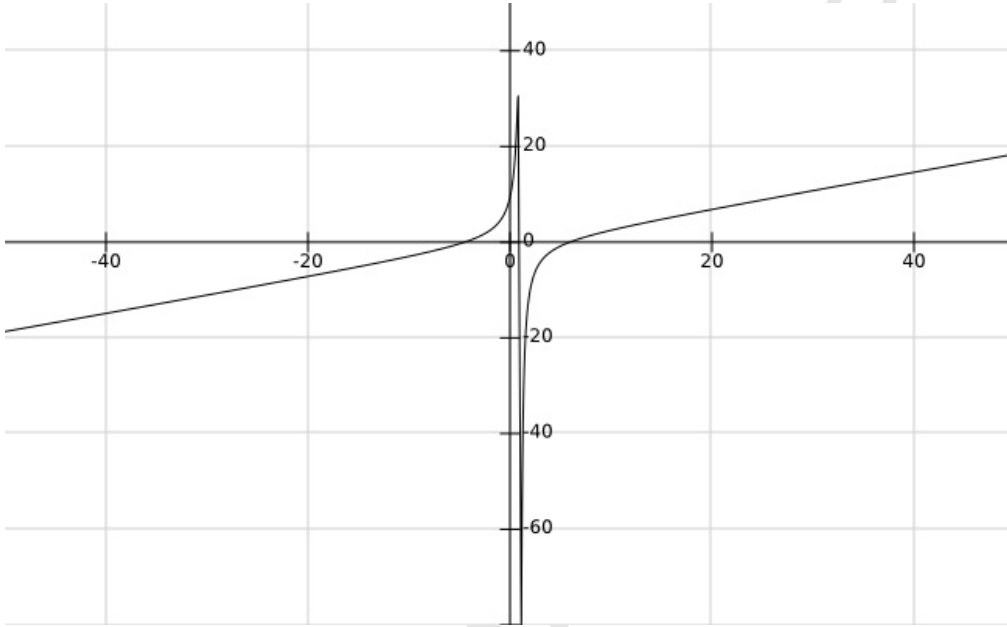


Figure 3: A plot between Y_3 (y axis) and G (x axis) when the initial condition is 1.

Initially we have to compute the determinants:

$$D_3^{(0)} = \begin{vmatrix} A_1^{(0)} & A_1^{(1)} & 0 \\ A_2^{(0)} & A_2^{(1)} & A_2^{(2)} \\ A_3^{(0)} & A_3^{(1)} & A_3^{(2)} \end{vmatrix} = \begin{vmatrix} \frac{\Gamma(0.5)}{\Gamma(-0.5)\Gamma(1)} & 1-G & 0 \\ \frac{\Gamma(0.5)}{\Gamma(1.5)} & \frac{\Gamma(0.5)}{\Gamma(-0.5)\Gamma(1)} & 1-G \\ \frac{\Gamma(0.5)}{\Gamma(2.5)} & \frac{\Gamma(0.5)}{\Gamma(-0.5)\Gamma(2)} & \frac{\Gamma(0.5)}{\Gamma(-0.5)\Gamma(1)} \end{vmatrix}.$$

By using the property that $\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) = (n-1)(n-2)(n-3)\Gamma(n-3)$, we get:

$$D_3^{(0)} = \begin{vmatrix} -0.5 & 1-G & 0 \\ -0.25 & -0.5 & 1-G \\ \frac{3}{16} & -0.25 & -0.5 \end{vmatrix} = -\frac{1}{8} - \frac{1}{4}(1-G) + \frac{3}{8}(1-G)^2.$$

Furthermore, $D_0^{(3)} = 1$, $D_1^{(2)} = A_1^{(2)} = \frac{1}{8}$ and

$$D_2^{(1)} = \begin{vmatrix} A_1^{(1)} & A_1^{(2)} \\ A_2^{(1)} & A_2^{(2)} \end{vmatrix} = \begin{vmatrix} 1-G & \frac{1}{8} \\ -0.5 & 1-G \end{vmatrix} = (1-G)^2 - \frac{1}{16}.$$

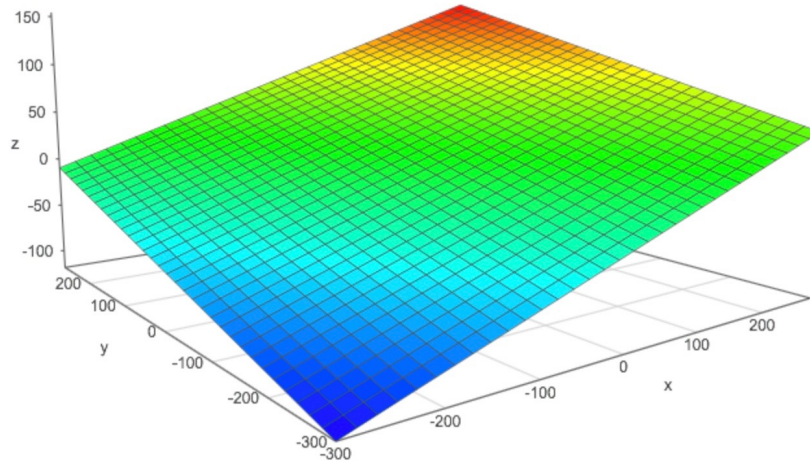


Figure 4: A 3D graph for Y_4 when the two initial conditions (bottom left and right axis respectively) take values in the interval $[-300, 300]$.

Hence

$$Y_3 = \left[-\frac{3}{8}(1-G) + \frac{1}{4} + \frac{1}{8}(1-G)^{-1}\right]Y_0 + \frac{1}{8}(1-G)^{-1} - \frac{1}{2}(1-G)^{-2} - \frac{1}{16}(1-G)^{-3}.$$

In Figures 1 and 2 we have graphs for Y_3 when G and the initial conditions take values in the interval $[-300, 300]$. For a given initial condition, for example $Y_0 = 1$ we get

$$Y_3 = -\frac{3}{8}(1-G) + \frac{1}{4} + \frac{81}{8}(1-G)^{-1} - \frac{1}{2}(1-G)^{-2} - \frac{1}{16}(1-G)^{-3}.$$

In Figure 3, we have a plot between Y_3 and G for this case.

Example 3.2.

We assume the following fractional difference equation for $k = 1, 2, \dots$

$$k\nabla_0^{\frac{1}{4}}Y_k = (k-1)Y_k + k^2Y_{k-1},$$

with $Y_k : \mathbb{N} \rightarrow \mathbb{C}$. The above equation belongs in the family of (2) for:

$$\begin{aligned} \alpha &= 0; \\ F_k &= k, \quad \forall k \in \mathbb{N}; \\ G_k^{(k)} &= k-1, \quad G_k^{(k-1)} = k^2 \quad \text{and} \quad G_k^{(j)} = V_k = 0, \quad \forall j \neq k, k-1; \\ A_k^{(k)} &= F_k - G_k^{(k)} = k - (k-1) = 1; \\ A_k^{(k-1)} &= \frac{1}{\Gamma(-0.25)} (2)^{-1.25} (F_k - G_k^{(k-1)}) = \frac{1}{\Gamma(-0.25)} (k - k^2) = \frac{\Gamma(0.75)}{\Gamma(1.25)\Gamma(-0.25)} k(1-k); \\ A_k^{(j)} &= \frac{1}{\Gamma(-0.25)} (k-j+1)^{-0.75} = \frac{\Gamma(k-j-0.5)}{\Gamma(-0.25)\Gamma(k-j)}, \quad \forall j \neq k, k-1. \end{aligned}$$

Since the system is homogeneous, from Corollary 2.1 there exists a unique solution given by (9):

$$Y_k = (-1)^k \frac{D_k^{(0)}}{k^{k-1}} Y_0$$

If the initial condition is given, for example $Y_0 = 1$ and we need the value Y_2 then:

$$Y_2 = \frac{1}{2} D_2^{(0)}.$$

Initially we have to compute the determinants:

$$D_2^{(0)} = \begin{vmatrix} A_1^{(0)} & A_1^{(1)} \\ A_2^{(0)} & A_2^{(1)} \end{vmatrix} = \begin{vmatrix} (1-1^2) \frac{\Gamma(0.75)}{\Gamma(-0.25)\Gamma(2)} & 1 \\ 2 \frac{\Gamma(1.75)}{\Gamma(-0.25)\Gamma(3)} & (-2) \frac{\Gamma(0.75)}{\Gamma(-0.25)\Gamma(2)} \end{vmatrix}.$$

By using the property $\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) = (n-1)(n-2)(n-3)\Gamma(n-3)$, we get:

$$D_2^{(0)} = \begin{vmatrix} 0 & 1 \\ -0.1875 & 0.5 \end{vmatrix} = 0.1875.$$

Hence

$$Y_2 = 0.09375.$$

Example 3.3.

We assume the following fractional difference equation for $k = 2, 3, \dots$ and $Y_k : \mathbb{N} \rightarrow \mathbb{C}$.

$$\nabla_0^{\frac{1}{2}} Y_k = 2^k.$$

From Corollary 2.2 there exists a unique solution given by (10). Where

$$\begin{aligned} \alpha &= 0; \\ F_k &= 1, \quad \forall k \in \mathbb{N}; \\ G_k^{(j)} &= 0 \quad \forall j, k \in \mathbb{N}; \\ A_k^{(k)} &= 1 \quad \text{and} \quad A_k^{(j)} = \frac{1}{\Gamma(-0.5)}(k-j+1)^{-1.5} = \frac{\Gamma(k-j-0.5)}{\Gamma(-0.5)\Gamma(k-j)}, \quad \forall j \neq k; \\ V_k &= 2^k; \end{aligned}$$

and then (10) takes the form

$$Y_k = (-1)^k [D_k^{(0)} Y_0 - D_{k-1}^{(1)} Y_1] + \sum_{i=0}^{k-2} (-1)^i D_i^{(k-i)} 2^{k-i}.$$

If we need for example the value Y_4 , then:

$$Y_4 = D_4^{(0)} Y_0 - D_3^{(1)} Y_1 + \sum_{i=0}^2 (-1)^i D_i^{(4-i)} 2^{4-i}.$$

We compute the determinants:

$$D_4^{(0)} = \begin{vmatrix} A_1^{(0)} & A_1^{(1)} & 0 & 0 \\ A_2^{(0)} & A_2^{(1)} & A_2^{(2)} & 0 \\ A_3^{(0)} & A_3^{(1)} & A_3^{(2)} & A_3^{(3)} \\ A_4^{(0)} & A_4^{(1)} & A_4^{(2)} & A_4^{(3)} \end{vmatrix} = \begin{vmatrix} \frac{\Gamma(0.5)}{\Gamma(-0.5)\Gamma(1)} & 1 & 0 & 0 \\ \frac{\Gamma(0.5)}{\Gamma(-0.5)\Gamma(2)} & \frac{\Gamma(0.5)}{\Gamma(-0.5)\Gamma(1)} & 1 & 0 \\ \frac{\Gamma(0.5)}{\Gamma(-0.5)\Gamma(3)} & \frac{\Gamma(0.5)}{\Gamma(-0.5)\Gamma(2)} & \frac{\Gamma(0.5)}{\Gamma(-0.5)\Gamma(1)} & 1 \\ \frac{\Gamma(0.5)}{\Gamma(-0.5)\Gamma(4)} & \frac{\Gamma(0.5)}{\Gamma(-0.5)\Gamma(3)} & \frac{\Gamma(0.5)}{\Gamma(-0.5)\Gamma(2)} & \frac{\Gamma(0.5)}{\Gamma(-0.5)\Gamma(1)} \end{vmatrix}.$$

By using $\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) = (n-1)(n-2)(n-3)\Gamma(n-3)$, we get:

$$D_4^{(0)} = \begin{vmatrix} -0.5 & 1 & 0 & 0 \\ -0.25 & -0.5 & 1 & 0 \\ \frac{3}{16} & -0.25 & -0.5 & 1 \\ -\frac{15}{96} & \frac{3}{16} & -0.25 & -0.5 \end{vmatrix} = 0.28125.$$

Furthermore,

$$D_3^{(1)} = \begin{vmatrix} A_2^{(1)} & A_2^{(2)} & 0 \\ A_3^{(1)} & A_3^{(2)} & A_3^{(3)} \\ A_4^{(1)} & A_4^{(2)} & A_4^{(3)} \end{vmatrix} = \begin{vmatrix} \frac{\Gamma(0.5)}{\Gamma(-0.5)\Gamma(1)} & 1 & 0 \\ \frac{\Gamma(0.5)}{\Gamma(-0.5)\Gamma(2)} & \frac{\Gamma(0.5)}{\Gamma(-0.5)\Gamma(1)} & 1 \\ \frac{\Gamma(0.5)}{\Gamma(-0.5)\Gamma(3)} & \frac{\Gamma(0.5)}{\Gamma(-0.5)\Gamma(2)} & \frac{\Gamma(0.5)}{\Gamma(-0.5)\Gamma(1)} \end{vmatrix},$$

or, equivalently,

$$D_3^{(1)} = \begin{vmatrix} -0.5 & 1 & 0 \\ -0.25 & -0.5 & 1 \\ \frac{3}{16} & -0.25 & -0.5 \end{vmatrix} = -0.1875.$$

Finally, $D_0^{(4)} = 1$, $D_1^{(3)} = A_4^{(3)} = \frac{\Gamma(0.5)}{\Gamma(-0.5)\Gamma(1)} = -0.5$ and

$$D_2^{(2)} = \begin{vmatrix} A_3^{(2)} & A_3^{(3)} \\ A_4^{(2)} & A_4^{(3)} \end{vmatrix} = \begin{vmatrix} \frac{\Gamma(0.5)}{\Gamma(-0.5)\Gamma(1)} & 1 \\ \frac{\Gamma(1.5)}{\Gamma(-0.5)\Gamma(2)} & \frac{\Gamma(0.5)}{\Gamma(-0.5)\Gamma(1)} \end{vmatrix} = \begin{vmatrix} -0.5 & 1 \\ -0.25 & -0.5 \end{vmatrix} = 0.5.$$

Hence

$$Y_4 = D_4^{(0)}Y_0 - D_3^{(1)}Y_1 + 2^2D_2^{(2)} - 2^3D_1^{(3)} + 2^4D_0^{(4)},$$

or, equivalently,

$$Y_4 = 0.28125Y_0 + 0.1875Y_1 + 22.$$

In Figures 4 we have a 3D graph between Y_4 and the two initial conditions.

Conclusions

In this article, we focused on generalized non-autonomous fractional nabla difference equations in the form of (2). This family of problems covers many linear fractional nabla difference equations studied by other authors. By using matrix theory we provide a new practical formula of solutions for (2), as well as for some alternative forms of this family of equations. Finally, we provide three different numerical examples to justify our theory.

As a further extension of this paper is to study the family of systems of non-autonomous fractional nabla difference equations as summarised in (2). For this study, very good knowledge of matrix pencil theory will be required since the coefficient can be non-square matrices or square and singular.

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