Robust analysis and synthesis design tools for digitally controlled power converters

Robust Analysis and Synthesis Design Tools for Digitally Controlled Power Converters

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***Abstract*—A new suite of Matlab-compatible robust analysis and synthesis design tools for digitally-controlled switched-mode power supplies has been developed. The objective of developing this tool suite is to assist control engineers to design and/or assess digital compensators that are *robustly* stable. The tool suite comprises of nonlinear, linearized and discrete-time continuous conduction mode and discontinuous conduction mode models, robust analysis algorithms and robust synthesis algorithms. To promote ease of use and adoption, a front-end graphical user interface has also been developed.**

I. Introduction

When developing compensators for switched-mode power supplies (SMPS), models of the SMPS are generally used. Quite often these compensators are developed using fixed nominal values for all the power rail components and parasitic estimates in the model description. This means there will be always gap between the model and the real SMPS as in reality there will be component or parameter variation, typically attributed to manufacturing tolerances, varying operating temperature conditions and/or age degradation. It therefore makes intuitive sense that being capable of incorporating these variations into the model will close the gap between the model and actual SMPS and therefore accommodate better compensator design which in turn will attribute to better performance. Conversely ignoring component variation could result in unacceptable degradation in performance and/or even instability over the operating lifetime of the SMPS. In control theory, incorporating variation into the model falls into the scope of "Robust Control", where two types of problems are generally considered: robust analysis problems and robust synthesis problems. In an effort to assist control engineers to develop more robust compensators and/or allow them to test how robust their existing designs are, a new suite of Matlab-compatible robust synthesis design and analysis tools for digitally-controlled SMPS has been developed.

II. Robust Control

The ability to incorporate component variation into the SMPS model closes the gap between the model dynamics and actual system behavior for predefined operating ranges. In robust control theory, this component variation is referred to as "uncertainty" and the theory has been well advanced over the last 30 years [1], [2]. Even though the theory and computations may seem relatively complex, it builds on intuitive blocks with every mathematical model being transformed into the same generic structure for both robust analysis and robust synthesis. To illustrate, consider the inductor L in a SMPS. For nominal valued-based models a specific value is assigned, for example $L = 500$nH. Incorporating component variation or uncertainty into the same model to reflect manufacturing tolerances, say $\pm 20\%$, can be mathematically described by equation (1)

$$L = L_0(1 + w_L\delta_L) \tag{1}$$

where L_0 is the nominal value, w_L is the weight or range of variation (in this case $w_L = 0.20$) and δ_L is an introduced normalized uncertain parameter where $|\delta_L| \leq 1$. Simple arithmetic confirms that this equation quantifies $L \in [400,\ 600]$nH noting also that if δ_L is set to zero, this returns the robust model description back to the nominal model description. Graphically, this can be represented by the block diagram description given in Figure 1. In order to build the robust model description, it is necessary to replace all of the nominal parameters with uncertainty representations similar to the one described in equation (1). This robust model can then be transformed into a generic $M - \Delta$ form as shown in Figure 2 where the Δ block contains all of the introduced normalized uncertain parameters. Transformation into this generic form can be performed mathematically, by block diagram and/or by using dedicated software tools [3]-[5].

When in this form, a frequency domain scalar metric called the "structured singular value", commonly abbreviated in the literature as μ, can be used to assess the robustness of any open- or closed-loop system. Put simply, for normalized parametric uncertainty, a value of $\mu_{\mathcal{K}}(M,\Delta) < 1$ mathematically verifies that the system is robustly stable for the actual or measured component and estimated parasitic variation (uncertainty). Conversely, $\mu_{\mathcal{K}}(M,\Delta) \geq 1$ confirms a system is not robustly stable. While this frequency domain metric is computationally intensive to calculate, it is easy to interpret. A valuable feature of some robustness analysis techniques is that candidate sets of parameter values contributing to worst-case stability and performance are returned. Also this same μ metric can be used for robust synthesis where (robustly) stable linear digital compensators are designed for predefined component and parasitic variation.

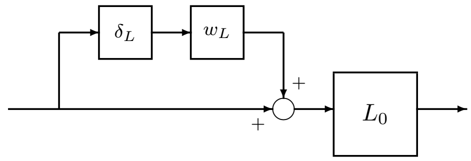

Fig. 1. Component uncertainty representation.

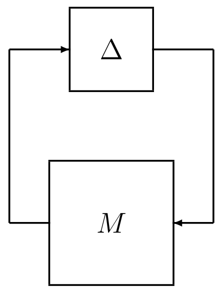

Fig. 2. System uncertainty representation.

III. Robust Analysis and Synthesis Design Tools

In an effort to assist control engineers to design more robust compensators (robust synthesis) and/or assess their existing designs (robust analysis), a complete suite of robust control design tools has been developed. The basic structure of the software design suite is shown in Figure 3 and comprises of SMPS Models, a robust analysis algorithm and a robust synthesis algorithm.

A. SMPS Models

A set of analytical small-signal state-space models for the buck, boost, buck-boost and flyback SMPS topologies have been developed. For each SMPS topology, non-linear continuous conduction mode (CCM) and discontinuous conduction mode (DCM) models, linearized CCM and DCM continuous-time and linear discrete-time CCM and DCM models have been derived. An important note is that for all model derivations circuit currents and voltages are averaged over the switching period. With reference to Figure 4, the state-space description for the CCM and DCM non-linear models is given by

$$\dot{x}(t) = f\left(x(t), u(t)\right) \quad (2a)$$
$$y(t) = g\left(x(t), u(t)\right) \quad (2b)$$

where $x(t)$ is the vector of converter states (e.g. inductor current and capacitor voltage, $x(t) = [\overline{i_L}(t), \overline{v_C}(t)]^T$), the input vector is $u(t) = [d(t), v_g(t)]^T$ (i.e. the duty cycle input and the input voltage) and the output vector $y = v_{out}(t)$, the output voltage. For each SMPS topology, hybrid/combined CCM-DCM models have been implemented in Simulink as shown in Figure 5. Linear analytical state-space models have

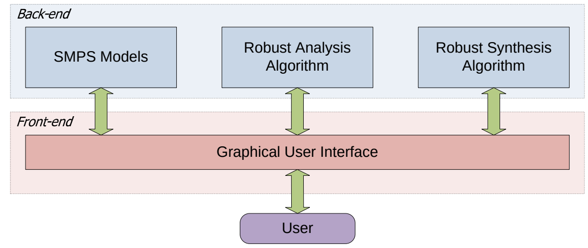

Fig. 3. Software design suite.

been derived via linearization of the nonlinear models at a particular operating point and are described by

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (3a)$$
$$y(t) = Cx(t) + Du(t) \quad (3b)$$

The input vector is now $u(t) = [\Delta d(t), v_g(t)]^T$ where $\Delta d(t) = d(t) - \mathcal{D}$ and $\mathcal{D}$ is the duty cycle quiescent operating point. The discrete-time CCM and DCM linear state-space models are derived similarly to how the discrete-time models detailed in [6] were derived. Due to the discrete nature of the averaged models, the sample time (T_s) is chosen to match the switching period. These derivations are based on the discrete-time state equation given by

$$x[n+1] = e^{AT_s}x[n] + \left(\int_0^{T_s} e^{A\nu}dv\right)Bu[n] \quad (4)$$

where $\nu = (n+1)T_s - \tau$. For computational reasons, the exponent term in (4) can be adequately approximated by

$$e^{AT_s} \approx I + AT_s \quad (5)$$

with the discrete-time state-space description given by

$$\begin{aligned} x[n+1] &= \underbrace{(I + AT_s)}_{\Phi}x[n] + \underbrace{(I + AT_s)BT_s}_{\Gamma}u[n] \\ y[n] &= Cx[n] + Du[n] \end{aligned} \quad (6)$$

All mathematical models have been validated using the commercially available PLECS toolset [7] for Simulink and allow for transient simulation of nonlinear and linearized averaged states-space models, where switching between the CCM and DCM models is performed by a mode switch logic (MSL) block. This MSL block is also shown in Figure 5 and was developed to enable smooth switching during transient simulations especially where transitions between the two conduction modes occur.

The analytical discrete-time state-space models are used as reference models for robust analysis and robust synthesis and have been further developed to incorporate component

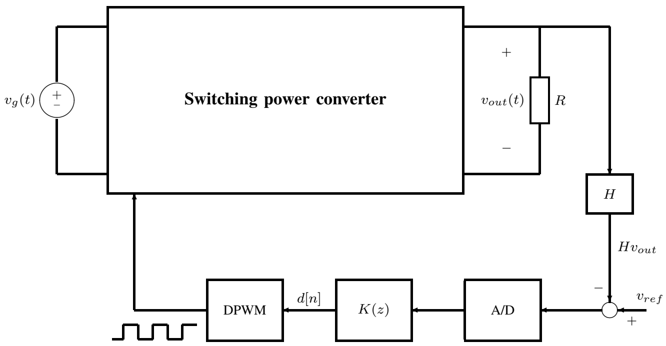

Fig. 4. SMPS with digital voltage-mode control.

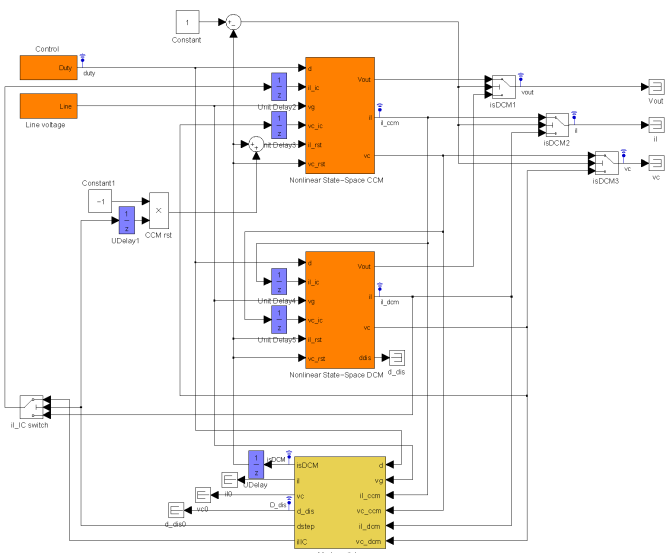

Fig. 5. SMPS CCM and DCM nonlinear models with MSL block.

and parasitic variation (uncertainty). In essence, all instances of the power rail and parasitic parameters within the state-space descriptions are replaced with expressions similar to (1). These descriptions in turn have been transformed into the generic upper linear fractional transformation (LFT) structure shown in Figure 6, noting that w is a vector signal including noise, disturbances and reference signals, and z is a vector signal including all controlled signals and tracking errors. If the matrix $\mathbf{M}$ in Figure 6 is partitioned as

$$\mathbf{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \tag{7}$$

then the mapping from $w \rightarrow z$ is given by

$$z = \left[M_{22} + M_{21} \Delta \left(I - M_{11} \Delta \right)^{-1} M_{12} \right] w \tag{8}$$

A case study example on the generation of discrete-time CCM and DCM models for a Buck converter is detailed in [8].

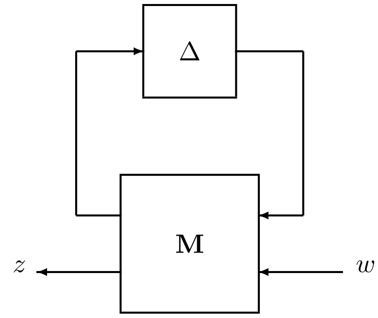

Fig. 6. Linear fractional transformation.

B. Robust Analysis

In the context of robust analysis, a control system is considered robust if it is insensitive to differences between the actual system (SMPS) and the model of the system which was used to design the compensator. The differences are referred to as model/plant mismatch or simply model uncertainty. The main idea in this robust control paradigm is to check whether the design specifications are satisfied even for the worst-case uncertainty [2]. For the SMPS topologies presented, strictly real parametric uncertainty is considered in the SMPS models. Uncertainty in this context is the actual or measured component and estimated parasitic variation.

As previously stated, a frequency domain scalar metric called the "structured singular value", μ, can be used to assess the robustness of any open- or closed-loop system. The formal definition of μ is [9]

The structured singular value, $\mu_{\mathcal{K}}(M)$, of a matrix $M \in \mathbb{C}^{n \times n}$ with respect to a block structure $\mathcal{K}(m_r, m_c, m_C)$ is then defined as

$$\mu_{\mathcal{K}}(M) = \frac{1}{\min\limits_{\Delta \in X_{\mathcal{K}}} \{ \overline{\sigma}(\Delta) : \det(I_n - \Delta M) = 0 \}} \tag{9}$$

with $\mu_{\mathcal{K}}(M) = 0$ if no $\Delta \in X_{\mathcal{K}}$ solves $\det(I_n - \Delta M) = 0$.

The generic M-Δ structure given in Figure 2 is the basis for this definition. It should also be noted that the uncertain parameters are strictly real valued for the class of problems of interest, therefore the set of allowable perturbations may be quantified formally as

$$X_{\mathcal{K}} = \left\{ \Delta = \text{block diag} \left(\delta_1^r I_{k_1}, \ldots, \delta_{m_r}^r I_{k_{m_r}} \right) : \delta_i^r \in \mathbb{R} \right\}$$

For this robust analysis, the SMPS feedback configuration shown in Figure 4 is the configuration of interest. Currently voltage mode control is only considered but it is planned to extend this analysis to current mode control as well. Using the mathematical techniques, block diagram algebra and/or software tools detailed in [1], [2], the SMPS feedback configuration shown in Figure 4 can be transformed into the generic LFT structure given in Figure 6. The final transformation to the M-Δ closed-loop feedback structure in Figure 2 necessary for μ-analysis becomes more apparent noting that for the formulations given, $M := M_{11}$. When in this form, the structured singular value μ can be computed.

One perceived drawback of any robust analysis using μ is the computation of μ itself as this computation is NP-hard, even for reasonably-sized problems [9]. Instead upper and lower bounds on μ are computed using polynomial-time algorithms. As part of this work, a new computationally efficient lower bound algorithm to compute μ for a specific compensator design and predefined uncertainty has been developed. The new algorithm has been implemented as a Matlab function (`mu_pm`) and can compute a lower bound on μ for strictly real uncertainty problems which are the class of problems of interest. The full details of this lower bound μ algorithm will be published in [10]. The process flow for the calculation of μ using this algorithm for digitally-controlled SMPS topologies is given in Figure 7.

As stated in the section II, even though computation of μ is relatively difficult, it is very easy to interpret as a value of $\mu_{\mathcal{K}}(M, \Delta) < 1$ verifies that the system is robustly stable for the predefined uncertainty, while $\mu_{\mathcal{K}}(M, \Delta) \geq 1$ confirms a system is not robustly stable. As the algorithm developed is a lower bound, candidate worst-case component and parasitic values are returned and therefore worst-case responses can be plotted and compared for example with the nominal response i.e. where only fixed (nominal) values of the component values and parasitic estimates are used. It should be noted that when calculating a lower bound estimate on μ, an upper bound estimate must also be calculated. Commercial software available such as the Matlab "Robust Control Toolbox User's Guide" [3] from Mathworks Ltd can readily perform this task. It should also be noted that the μ algorithms implemented as part of this Matlab toolbox generally return poor estimates on the lower bound on μ for strictly real parametric uncertainty, whereas the lower algorithm presented in [10] generally returns good estimates on μ for this class of problem.

C. Robust Synthesis

With reference to the closed-loop configuration in Figure 8, robust synthesis allows the control engineer to develop a digital compensator $\mathbf{K} = K(z)$ for a specific SMPS topology and predefined component variations. The general synthesis problem becomes that of finding a compensator $\mathbf{K}$ achieving

$$\inf_{\mathbf{K} \in \mathcal{X}_S} \sup_{\omega \in \mathbb{R}^e} \mu_{\mathcal{K}} \left(\mathbf{M}(\mathbf{P}, \mathbf{K})(\mathbf{j}\omega) \right) \tag{10}$$

where $\mathcal{X}_S$ denotes all $\mathbf{K}$ that render $\mathbf{M}$ internally stable i.e. the set of *nominally* stabilizing compensators. Linear robust compensators are fixed and time-invariant and therefore can be readily implemented for digitally-controlled switching converters. A new μ synthesis algorithm that employs the lower bound `mu_pm` algorithm was developed. Due to the non-convex nature of the μ synthesis problem, non-gradient optimization was used to locate an optimal/suboptimal compensator. Currently compensator designs up to 4^{th} order are accommodated where

TABLE I. COMPENSATOR $K(z)$ ORDER AND DESIGN PARAMETERS.

$K(z)$ Order	Poles and Zeros Design Criteria	Optimization Variables
1	1 real	2
2	1 complex conjugate pair	4
3	1 real and 1 complex conjugate pair	6
4	2 complex conjugate pairs	8

$$K(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + b_4 z^{-4}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + a_4 z^{-4}} \tag{11}$$

This μ synthesis algorithm aims to provide optimum robustness for all allowable component variations by tuning the compensator coefficients. Initial constraints on the location of the compensator poles and zeros locations for each order are provided in Table I noting that there is a direct correlation between the coefficients and the number of optimization variables. As good design practice, all candidate digital compensators returned from the optimization algorithm are tested with the nonlinear CCM and DCM Simulink models. The proposed process flow for determining a candidate compensator $K(z)$ for a specific SMPS topology using the `mu_pm` algorithm is given in Figure 9. It is the intention to publish this robust synthesis algorithm in due course. Interested readers are referred to [1], [2] as excellent texts on this subject.

IV. GRAPHICAL USER INTERFACE

For ease of use, a Matlab-compatible graphical user interface has been developed for the robust design tools. The menu layouts for robust analysis and robust synthesis are shown in Figure 10 and Figure 11 respectively and was designed to provide all toolset functionality in a user-friendly professional way. The interface is simple to use where the SMPS topology can be selected and the nominal value and variation tolerances can be set. Depending on the model type other necessary information may also be required, namely switching frequency, sample period and the operating point. For robust analysis, a fixed structure compensator design needs to be chosen. Also the graphical user interface allows the user to graph time and frequency plots to compare nominal and worst case responses. The graphical user interface has the capability to load SMPS with controller configurations from a file while also allowing configurations to be saved to a file. PDF reports that include numerical results and graphs from robustness analysis or synthesis can also be generated.

V. CONCLUSION

Details of robust analysis and synthesis design tools for digitally-controlled SMPS have been presented. The analysis tools have been developed to assess the robustness of closed-loop designs where the SMPS power rail components and parasitics may be subject to variation. Using the synthesis tools, robust linear digital compensators can be automatically designed. A front-end graphical user interface has also been developed for ease of use and adoption. Buck, Boost, Buck-Boost and Flyback SMPS models have been developed with future work focused on developing robust models for other converter topologies.

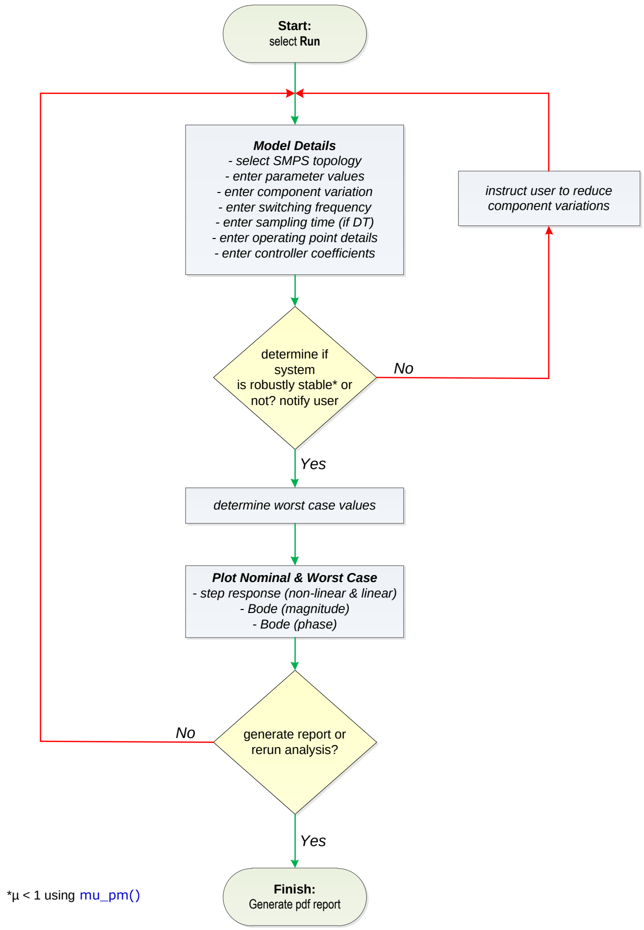

Fig. 7. Robust analysis process flow.

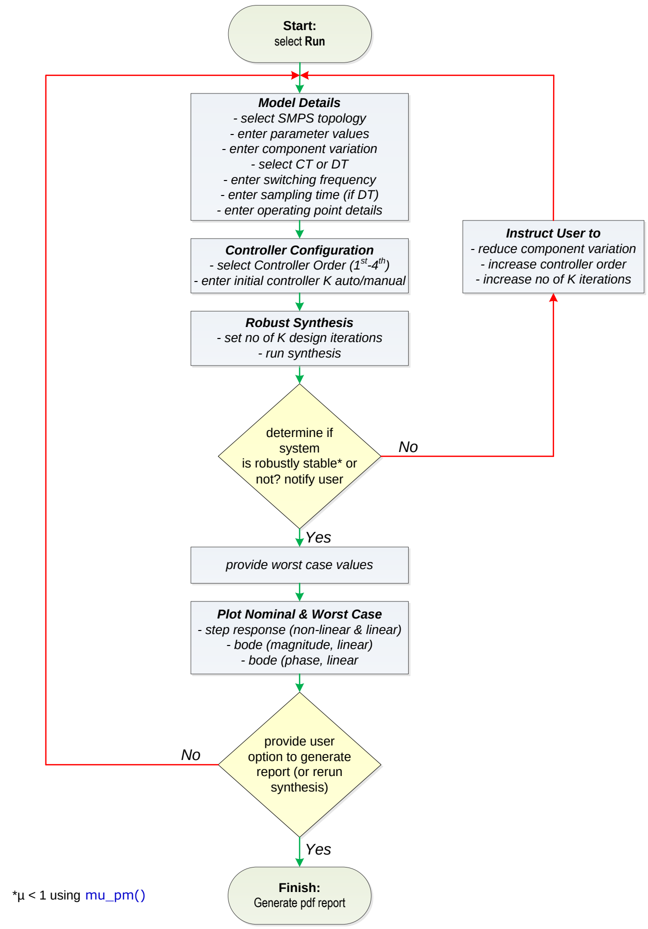

Fig. 9. Robust synthesis process flow.

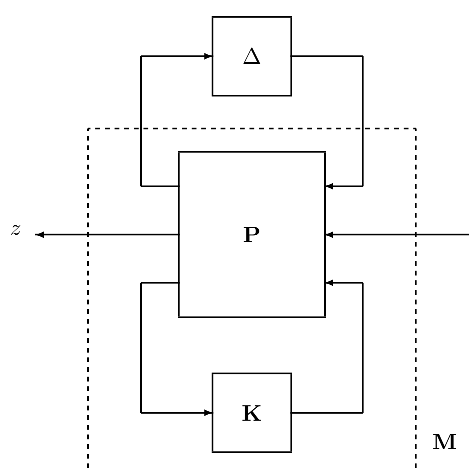

Fig. 8. General control and proposed μ synthesis controller design framework.

ACKNOWLEDGMENT

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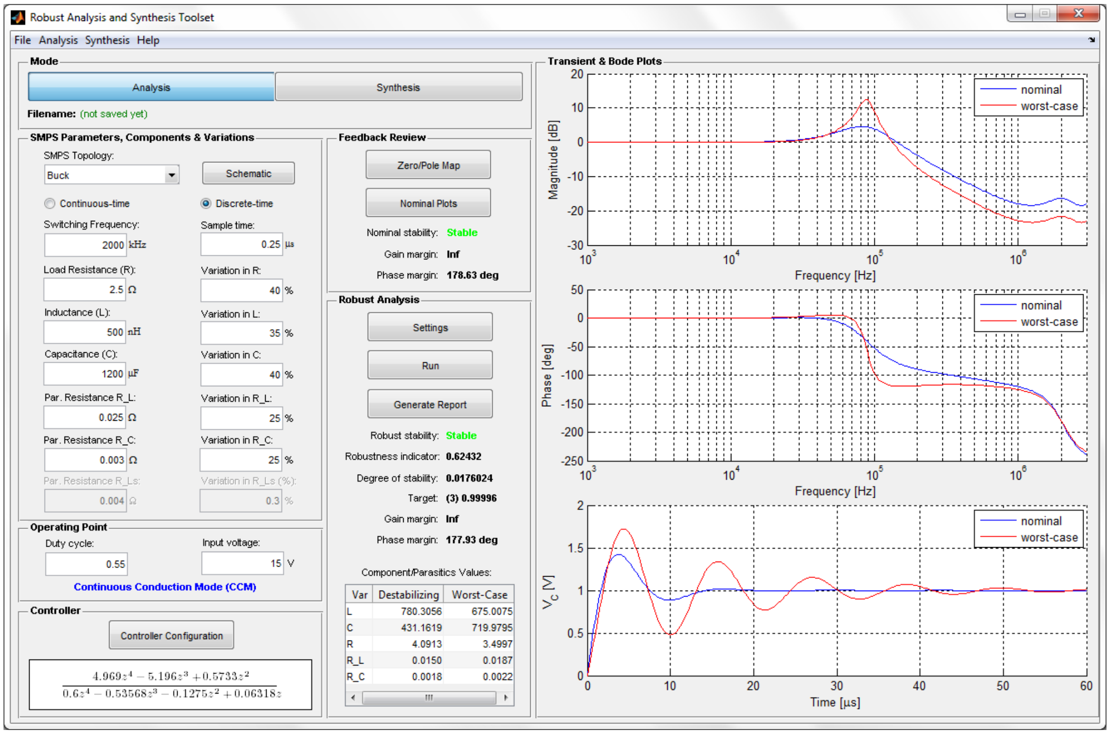

Fig. 10. Robust analysis menu layout.

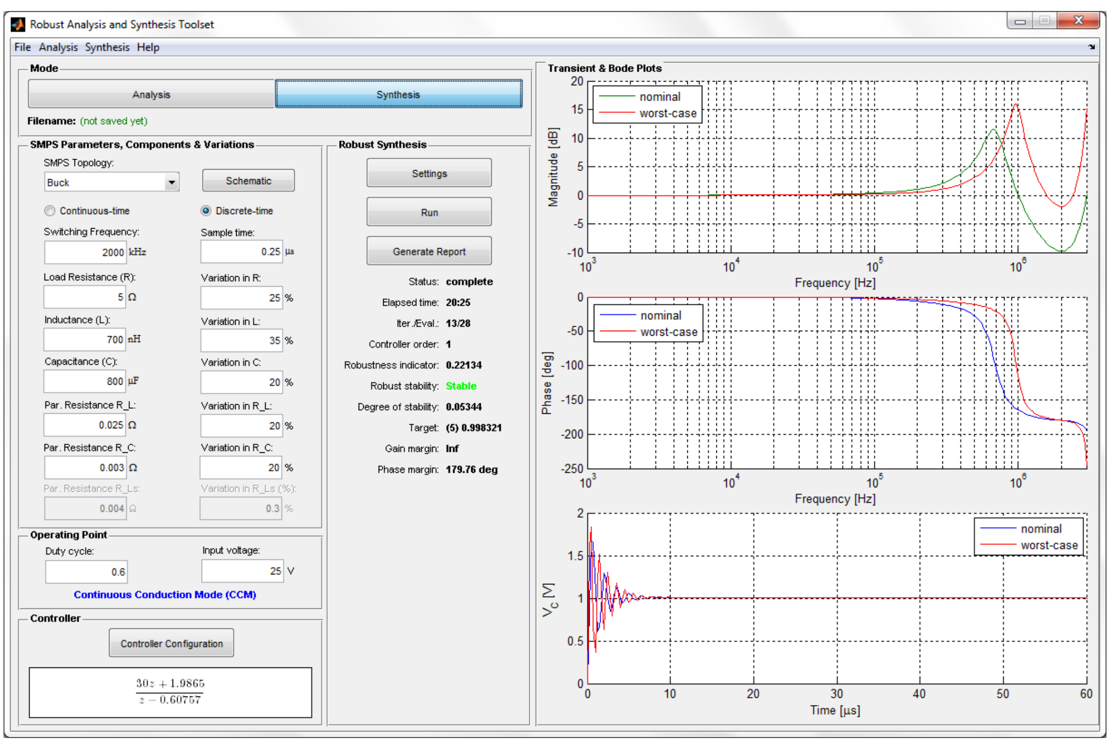

Fig. 11. Robust synthesis menu layout.