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Closed-form solutions for the coupled deflection of anisotropic Euler–Bernoulli composite beams with arbitrary boundary conditions

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ABSTRACT

The fully anisotropic response of composite beams is an important consideration in diverse applications including aeroelastic responses of helicopter rotor and wind turbine blades. Our goal is to present exact analytical solutions for the first time for coupled deflection of Euler–Bernoulli composite beams. Towards this goal, two approaches are proposed: (1) obtaining the exact analytical solutions directly from the governing equations of Euler–Bernoulli composite beams and (2) extraction of the solutions from Timoshenko composite beam solutions. For the direct solution approach, based on Euler–Bernoulli theory, new variationally-consistent field equations are obtained, in which four degrees of freedom, i.e. in-plane bending, out-of-plane bending, twist and axial elongation are fully coupled. By expressing the coupled system of differential equations in a compact matrix form, a novel expression for the eccentricity of neutral axes from the midplane, as well as the shift in shear centre from the centre of beam, is obtained. This eccentricity matrix serves to decouple the bending in the two principal directions from in-plane and twist deformations. Then, the general closed-form analytical solutions for the decoupled system are derived simply using direct integration. Additionally, the analogous closed-form analytical solutions are retrieved from the previously obtained Timoshenko composite beam solution and it is proven that they are identical to those obtained from the current direct approach for conditions where Euler–Bernoulli beam theory apply. To study the effects of anisotropy, numerical results are obtained for a number of examples with different composite stacking sequences showing various coupled behaviours. The results are compared against the Chebyshev collocation method as well as against less comprehensive analytical solutions available in the literature, noting that excellent agreement is observed, where expected. The present exact solutions can serve as benchmark problems for assessing the accuracy and convergence of various analytical and numerical methods.

1. Introduction

Beam theories are widely used by aerospace, biomedical, civil and other engineers as reduced-order mathematical models. Due to the continuous growth of composite materials usage, there is an increasing interest in composite beam problems, in particular, in their static response, which is of fundamental engineering importance. Any structural beam-like element considered in engineering designs must be sufficiently strong to withstand various types of loads and deflect within a prescribed range to meet design requirements and functional needs. Therefore, static analysis of beams, including deflection and cross-sectional stress calculations, is an important concept in engineering design and analysis and a subject of intense interest from the scientific community. Analytical solutions offer reliable and efficient tools for in-depth studies of the effects of different physical parameters while they are free from computational instabilities compared to numerical methods. In addition, they can be used as benchmarks to study the convergence and accuracy of numerical solutions. Different types of solutions, both analytical and semi-analytical, have been proposed by researchers for the static analysis of composite beams. Lekhnitskii [1] presented an exact solution for laminated beams using Airy stress polynomial functions by assuming that each layer is specially orthotropic in the plane of bending. Khdeir and Reddy [2] provided the exact closed-form solutions for the in-plane static analysis of symmetric and antisymmetric cross-ply laminated composite beams with arbitrary boundary conditions using classical, first-order, second-order and third-order shear deformation theories. The state space concept in combination with the Jordan canonical form was used to solve the governing equations. Rand [3] provided strength-of-materials type closed-form analytical solutions for the displacements of orthotropic composite

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beams with arbitrary solid cross-sections and general thin-walled geometry. Considering different types of loading, Rand modelled beams with a three-dimensional warping function with various basic composite beam configurations. Later, Rovenski and Rand [4] formulated an analytical solution for the elastic response of anisotropic composite beams with rectangular cross-section based on the expression of the stress tensor components as trigonometric series and exponential functions. Using anisotropic elasticity theory, Esendemir et al. [5] obtained an analytical solution for the deflection problem of simply supported composite beams subject to a linearly distributed load. Ghugal and Shinde [6] used layerwise trigonometric shear deformation theory to obtain a closed-form solution for the bending problem of two-layer cross-ply laminated simply supported beams subject to sinusoidal load. Based on a higher-order theory which accounts for a higher-order variation of the axial displacement, Nguyen et al. [7] presented an analytical trigonometric series solution for the static, buckling and vibration analysis of laminated composite beams with arbitrary lay-ups. Considering the effect of strain gradient elasticity, Sidhardh and Ray [8] derived exact solutions for the static bending response of simply-supported isotropic and orthotropic laminated beams subject to a sinusoidally distributed load.

Some of the researchers focused solely on thin-walled beam problems. Based on the variational asymptotic analysis of two-dimensional shell theory, Berdichevsky et al. [9] introduced closed-form expressions for the stiffness coefficients and the displacement and stress fields of these types of composite beams made of anisotropic materials. Song et al. [10] presented analytical solutions for the static response of composite I-beams loaded at their free-end cross-section. Jung and Lee [11] presented an analytical closed-form solution for the static behaviour of both symmetric and anti-symmetric thin-walled composite I-beams with transverse shear couplings subject to unit bending or torque load at the beam tip. The beams were considered to be fixed at one end and the other end was restrained against warping. Using a Taylor series expansion, Pluzsik and Kollár [12] proposed an exact analytical solution for simply supported thin-walled orthotropic beams with closed cross-section subject to a distributed sinusoidal torque load about the beam reference axis. Kim et al. [13] used Vlasov's assumptions to derive exact solutions for the analysis of thin-walled anisotropic composite beams subject to torsional moments in which the beam cross-section is symmetrically laminated. Clamped–clamped, clamped–hinged and hinged–hinged boundary conditions were considered. Kim and Lee [14] provided power series solutions for the coupled static analysis of thin-walled laminated monosymmetric I-section beams resting on an elastic foundation. Vukasović et al. [15,16] developed closed-form analytical solutions based on Vlasov's theory with the influence of shear for the bending and torsion of thin-walled laminated beams, respectively. Simply supported and clamped orthotropic beams with symmetrical lay-up and open symmetrical and unsymmetrical cross-sections were considered. Kim and Lee [17] derived exact solutions for the decoupled deflections of thin-walled sandwich beams with material properties graded through the wall thickness and with non-symmetric cross-sections with three types of boundary conditions: simply supported, clamped and clamped–free.

In the context of sandwich beams problem, Pagani et al. [18] obtained a closed-form solution based on the unified formulation in which Layer-Wise (LW) Lagrange polynomials are used to express the three-dimensional (3D) displacement field via arbitrary order approximation of pure displacement variables in each layer over the cross-section. Subsequently, they applied this method to the static response analyses of cross-ply laminated and sandwich beams with simply supported boundary conditions. Recently, Yan et al. [19] presented a closed-form Navier-type solution using various higher-order beam theories for the static analysis of laminated fibre-reinforced and sandwich beams with simply supported boundary conditions. The higher-order beam models were developed by applying the unified formulation such that Lagrange-polynomials expansions of various orders approximate the kinematic field over the cross-section.

While Doeva et al. [20] presented exact solutions for composite beams under non-uniformly distributed loads, due to the best authors' knowledge, there is no exact solution for a fully coupled 3D Euler–Bernoulli composite beam subject to uniform and tip loads in the literature. It is shown that the solutions obtained for the uniform loads in this work are not the special cases of non-uniformly distributed loads and thus cannot be extracted from those of Doeva et al. [20]. Recently, Doeva et al. [21] obtained the exact analytical solutions for fully coupled Timoshenko composite beams. To obtain the exact solutions, six coupled governing equations were expressed in matrix form with internal forces and moments being decoupled. Then, using the direct integration technique, the exact solutions for the vectors of rotations and displacements of the beam were derived. However, in the case of Euler–Bernoulli theory, since rotations of the beam are expressed in terms of transverse displacements, it is not possible to employ the same strategy as for Timoshenko theory [21] to obtain the exact solutions for fully coupled Euler–Bernoulli composite beams, noting it is not yet available.

Therefore, in order to fill this important lack of knowledge, the main purpose of current work is to obtain an exact analytical solution for the coupled static deflection of Euler–Bernoulli composite beams with constant stiffness. To directly obtain the exact solutions from the governing equations, as opposed to the approach taken in [21], an eccentricity matrix is obtained in the current work to decouple bending in the two principal directions from axial elongation and twist. It is also shown that the exact solutions of Euler–Bernoulli composite beams can be retrieved from those of Timoshenko composite beams [21] by neglecting transverse shear effects. However, the equivalence of the corresponding coefficient matrices appearing in the solutions obtained from the two approaches is not directly obvious and cannot be considered a priori. Therefore, herein it is proven that the correlated coefficient matrices from both approaches are identical, consequently, it is shown that the solutions are therefore identical. This proof verifies both approaches and demonstrates they are consistent with the fundamental concepts of Timoshenko and Euler–Bernoulli beam theories. It is also important to note the contribution of Masjedi and Weaver [22,23] who presented approximate series solutions for variable stiffness Euler–Bernoulli composite beams. For variable stiffness composite beams the governing differential equations have arbitrary variable coefficients and it is not possible to obtain exact solutions for a general distribution of stiffness properties [22,23]. In contrast to variable stiffness beams, the coefficients in the differential terms of governing equations are constant in constant stiffness beams and exact solutions can be obtained. It is noted that the closed-form exact solutions obtained in the current work cannot be retrieved from approximate series solutions proposed previously by Masjedi and Weaver [22]. In order to further clarify the new contributions of the current work, a comparison is made between the available analytical solutions for composite beams in Tables 1 and 2.

While the focus of this paper is to present an exact analytical solution for the static response of fully coupled composite beams, for the purpose of verification the Chebyshev collocation method, which has been shown to be efficient and accurate in beam problems [24,25], is also applied.

A brief outline of the rest of this paper follows: the governing equations of a fully coupled composite beam are presented in Section 2; in Section 3, a general closed-form analytical solution is obtained in a compact matrix form. Then, to obtain the expressions for the constants of integration, various boundary conditions such as clamped–free, clamped–clamped, clamped–simply supported and simply supported–simply supported subject to point and distributed loads, are applied. In Section 4, as an alternative approach, the solution is retrieved from the previously obtained Timoshenko composite beam solution and it is proven that it is identical to those obtained from the direct approach. In Section 5, several benchmark test problems are considered and the exact results are verified by other analytical solutions and those obtained from the Chebyshev collocation method. Some conclusions and remarks are provided in the last section.

Table 1
Comparison of available analytical solutions for composite beams.

	Theory				Kinematics		Cross-section		Coupling terms			Solution	
	EBT	**TBT**	**HOT**	**ET**	**2D**	**3D**	**SO**	**TW**	**BT**	**AT**	**BA**	**S**	**E/C**
Lekhnitskii [1]	–	–	–	✓	✓	–	✓	–	–	–	✓	–	✓
Khdeir and Reddy [2]	✓	✓	✓	–	✓	–	✓	–	–	–	✓	–	✓
Rand [3]	–	–	–	✓	–	✓	✓	✓	✓	✓	✓	–	✓
Rovenski and Rand [4]	–	–	–	✓	–	✓	✓	–	✓	✓	✓	✓	–
Song et al. [10]	–	✓	–	–	–	✓	–	✓	✓	✓	–	–	✓
Jung and Lee [11]	–	✓	–	–	–	✓	–	✓	✓	✓	–	–	✓
Esendemir et al. [5]	–	–	–	✓	✓	–	✓	–	–	–	✓	–	✓
Pluzsik and Kollár [12]	–	✓	–	–	–	✓	–	✓	✓	–	–	–	✓
Kim et al. [13]	✓	–	–	–	–	✓	–	✓	✓	–	–	–	✓
Ghugal and Shinde [6]	–	–	✓	–	✓	–	✓	–	–	–	✓	–	✓
Kim and Lee [14]	✓	–	–	–	–	✓	–	✓	✓	✓	✓	✓	–
Nguyen et al. [7]	–	–	✓	–	✓	–	✓	–	–	–	✓	✓	–
Vukasović et al. [15,16]	–	–	–	✓	–	✓	–	✓	✓	–	–	–	✓
Kim and Lee [17]	✓	–	–	–	–	✓	–	✓	–	–	–	–	✓[a]
Pagani et al. [18]	–	–	✓	✓	–	✓	✓	✓	✓	✓	✓	✓	–
Yan et al. [19]	–	–	✓	✓	–	✓	✓	✓	✓	✓	✓	✓	–
Sidhardh and Ray [8]	–	–	–	✓	✓	–	✓	–	–	–	✓	–	✓
Present	✓	–	–	–	–	✓	✓	✓	✓	✓	✓	–	✓

EBT: Euler–Bernoulli Beam Theory
TBT: Timoshenko Beam Theory
HOT: Higher Order Beam Theory
ET: 2D/3D Elasticity Theory
SO: Solid Cross-Section
TW: Thin-Walled Cross-Section
BT: Bend–Twist
AT: Axial Elongation–Twist
BA: Bend–Axial Elongation
S: Series Solution
E/C: Exact/Closed-Form.
[a]Closed-form expressions are only provided for decoupled beams.

Table 2
Comparison of available closed-from solutions for fully coupled composite beams.

	Theory		Stiffness		Loading type			Solution	
	EBT	**TBT**	**Constant**	**Variable**	**UD**	**NUD**	**CT**	**Series**	**Exact**
Masjedi and Weaver [22]	✓	–	–	✓	✓	–	✓	✓	–
Masjedi and Weaver [23]	✓	–	–	✓	–	✓	–	✓	–
Doeva et al. [20]	✓	–	✓	–	–	✓	–	–	✓
Doeva et al. [21]	–	✓	✓	–	✓	–	✓	–	✓
Present	✓	–	✓	–	✓	–	✓	–	✓

EBT: Euler–Bernoulli Beam Theory
TBT: Timoshenko Beam Theory
UD: Uniformly Distributed Load
NUD: Non-Uniformly Distributed Load
CT: Concentrated/Tip Load.

2. Governing equations

2.1. Beam kinematics

The mathematical model of the fully coupled Euler–Bernoulli composite beam presented in this paper is based on the following assumptions:

1. Material is linearly elastic.
2. Displacements, strains and rotations are small.
3. Cross-section is rigid, meaning in-plane or out-of-plane warping deformations are not considered.
4. Non-classical effects such as transverse shear and Vlasov effect are ignored.
5. The model is globally three-dimensional, i.e. all four degrees of freedom, namely axial displacement, twist, out-of-plane bending and in-plane bending, are taken into account.

Consider a straight composite beam for which its length ℓ is measured along the x coordinate axis while the coordinates y and z define the cross-sectional planes (see Fig. 1).

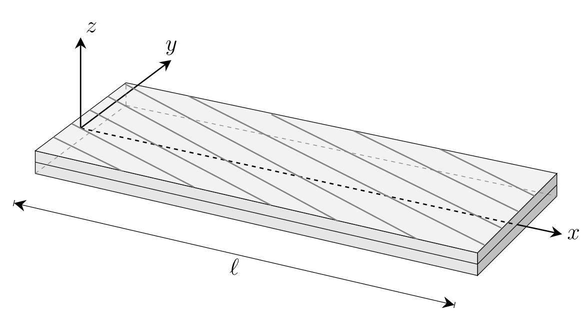

Fig. 1. Coordinate system for the composite beam.

The displacement vector $\boldsymbol{U} = \boldsymbol{U}(x, y, z)$ of a generic point on the cross-section is given as:

$$U_x = u(x) + z\theta_y(x) - y\theta_z(x), \tag{2.1a}$$

$$U_y = v(x) - z\varphi(x), \tag{2.1b}$$

$$U_z = w(x) + y\varphi(x), \tag{2.1c}$$

where U_x, U_y and U_z are the components of displacement vector $\boldsymbol{U}$ and u, v and w denote displacements of the beam reference line in x, y and z directions and φ, θ_y and θ_z are the rotations of beam cross-section about x, y and z, respectively. It is worth noting that the beam is assumed to be sufficiently slender that the displacements of the beam reference line and the rotations of the beam cross-section are only functions of x.

Now with the help of Eqs. (2.1) the strain measures of the beam are expressed as:

$$\epsilon_{xx} = \frac{\partial U_x}{\partial x} = u' + z\theta'_y - y\theta'_z, \tag{2.2a}$$

$$\gamma_{xy} = \frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x} = \left(v' - \theta_z\right) - z\varphi', \tag{2.2b}$$

$$\gamma_{xz} = \frac{\partial U_x}{\partial z} + \frac{\partial U_z}{\partial x} = \left(w' + \theta_y\right) + y\varphi', \tag{2.2c}$$

in which ϵ_{xx} is the axial strain, γ_{xy} and γ_{xz} are the transverse shear strains and ($'$) denotes derivative with respect to x. From Eqs. (2.1), other strain measures ϵ_{yy}, ϵ_{zz} and γ_{yz} are readily determined to be zero. According to Euler–Bernoulli beam kinematical relations which assume that the cross-section remains orthogonal to the beam reference axis after deformation, then:

$$\theta_y = -w', \quad \theta_z = v'. \tag{2.3}$$

2.2. Principle of virtual work

Internal work of the beam can be expressed as:

$$\int_V W_{int} dV = \int_V \left(\sigma_{xx}\epsilon_{xx} + \sigma_{xy}\gamma_{xy} + \sigma_{xz}\gamma_{xz}\right) dV, \tag{2.4}$$

where σ_{xx} is the axial stress and σ_{xy} and σ_{xz} are the transverse stress distributions.

By substituting strain measures ϵ_{xx}, γ_{xy} and γ_{xz} given by Eqs. (2.2) into Eq. (2.4), we obtain:

$$\int_V W_{int} dV = \int_V \Big(\sigma_{xx}u' + z\sigma_{xx}\theta'_y - y\sigma_{xx}\theta'_z + \sigma_{xy}\left(v' - \theta_z\right) + \sigma_{xz}\left(w' + \theta_y\right) + \left(y\sigma_{xz} - z\sigma_{xy}\right)\varphi'\Big) dV. \tag{2.5}$$

Defining beam internal forces and moments as:

$$F_x = \int_A \sigma_{xx} dA, \tag{2.6a}$$

$$M_x = \int_A \left(y\sigma_{xz} - z\sigma_{xy}\right) dA, \tag{2.6b}$$

$$M_y = \int_A z\sigma_{xx} dA, \tag{2.6c}$$

$$M_z = -\int_A y\sigma_{xx} dA, \tag{2.6d}$$

and considering Eq. (2.3), then:

$$\int_0^\ell W_{int} dx = \int_0^\ell \left(F_x u' + M_x\varphi' + M_y\theta'_y + M_z\theta'_z\right) dx = \int_0^\ell \left(F_x u' + M_x\varphi' - M_y w'' + M_z v''\right) dx, \tag{2.7}$$

where F_x is the generalised axial force, M_x is the generalised twist moment, and M_y and M_z are the generalised bending moments.

The principle of virtual work is expressed as:

$$\int_0^\ell \left(\delta W_{int} - \delta W_{ext}\right) dx = 0, \tag{2.8}$$

where δW_{int} and δW_{ext} are the variations of internal and external works respectively. Variation of internal work of the beam can be derived from Eq. (2.7) as:

$$\int_0^\ell \delta W_{int} = \int_0^\ell \delta\boldsymbol{\epsilon}^T \boldsymbol{N} dx = \int_0^\ell \delta\boldsymbol{\epsilon}^T \boldsymbol{S}\boldsymbol{\epsilon} dx \tag{2.9}$$

where vector of strains $\boldsymbol{\epsilon}$, vector of internal forces and moments $\boldsymbol{N}$ and stiffness matrix $\boldsymbol{S}$ can be written as:

$$\boldsymbol{\epsilon} = [\epsilon_x \quad \kappa_x \quad \kappa_y \quad \kappa_z]^T = [u' \quad \varphi' \quad -w'' \quad v'']^T, \tag{2.10a}$$

$$\boldsymbol{N} = [F_x \quad M_x \quad M_y \quad M_z]^T, \tag{2.10b}$$

$$\boldsymbol{S} = \begin{bmatrix} EA & S_{ET} & S_{EF} & S_{EL} \\ S_{ET} & GJ & S_{FT} & S_{LT} \\ S_{EF} & S_{FT} & EI_y & S_{FL} \\ S_{EL} & S_{LT} & S_{FL} & EI_z \end{bmatrix}, \tag{2.10c}$$

where EA, GJ, EI_y and EI_z, are the extensional, twist, out-of-plane bending and in-plane bending stiffness respectively and coupling terms S_{ET}, S_{EF}, S_{EL}, S_{FT}, S_{LT} and S_{FL} are the coupling between axial elongation and twist, out-of-plane bending and axial elongation, in-plane bending and axial elongation, out-of-plane bending and twist, in-plane bending and twist, and out-of-plane and in-plane bending, respectively. It is worth mentioning that herein a linear relation is assumed between internal forces and moments $\boldsymbol{N}$ and strains and curvatures $\boldsymbol{\epsilon}$, i.e. $\boldsymbol{N} = \boldsymbol{S}\boldsymbol{\epsilon}$. Various approaches are proposed in the literature to obtain this linear relation for composite beams with arbitrary cross-section in order to extract stiffness matrix $\boldsymbol{S}$, e.g. see [26–31]. The so called linear 2D cross-sectional analysis is treated as a decoupled problem from 1D beam analysis. This approach can be traced back to Berdichevskii [32] who proposed that the 3D problem of a beam can split into a "2D cross-sectional analysis" governing the cross-sectional elastic constants and a "1D beam" analysis. Consequently, all entries of the stiffness matrix $\boldsymbol{S}$ are defined by symbolic engineering constants to keep the formulation as general as possible, while any suitable cross-sectional analysis tool can be employed to obtain numerical values for stiffness matrix entries. However, since in this work the 2D cross-sectional analysis and 1D beam problem are decoupled as proposed by Berdichevskii [32], the validity of exact closed-form solutions which are presented for a 1D beam problem in this paper are independent of the methodology used to obtain the stiffness matrix.

Variation of external work can be described by:

$$\int_0^\ell \delta W_{ext} dx = \int_0^\ell \delta\overline{\boldsymbol{U}}^T \boldsymbol{Q} dx, \tag{2.11}$$

where displacement vector $\overline{\boldsymbol{U}}$ and vector of external loads $\boldsymbol{Q}$ are written as:

$$\overline{\boldsymbol{U}} = [u \quad \varphi \quad w \quad v]^T, \tag{2.12a}$$

$$\boldsymbol{Q} = [q_x \quad q_\varphi \quad q_z \quad q_y]^T, \tag{2.12b}$$

where q_x, q_y, q_z are the distributed loads in the x, y, z directions respectively, and q_φ is the distributed torque.

Eqs. (2.9) and (2.11) can be written explicitly as:

$$\begin{aligned} \int_0^\ell \delta W_{int} dx = \int_0^\ell \{ & \delta u'\left(EAu' + S_{ET}\varphi' - S_{EF}w'' + S_{EL}v''\right) \\ & +\delta\varphi'\left(S_{ET}u' + GJ\varphi' - S_{FT}w'' + S_{LT}v''\right) \\ & -\delta w''\left(S_{EF}u' + S_{FT}\varphi' - EI_y w'' + S_{FL}v''\right) \\ & +\delta v''\left(S_{EL}u' + S_{LT}\varphi' - S_{FL}w'' + EI_z v''\right)\} dx, \end{aligned} \tag{2.13}$$

and

$$\int_0^\ell \delta W_{ext} dx = \int_0^\ell \left(\delta u q_x + \delta\varphi q_\varphi + \delta w q_z + \delta v q_y\right) dx. \tag{2.14}$$

Substituting Eqs. (2.13) and (2.14) into Eq. (2.8), the set of governing equations is derived as:

$$\begin{aligned} -EAu'' - S_{ET}\varphi'' + S_{EF}w''' - S_{EL}v''' &= q_x \\ -S_{ET}u'' - GJ\varphi'' + S_{FT}w''' - S_{LT}v''' &= q_\varphi \\ -S_{EF}u''' - S_{FT}\varphi''' + EI_y w^{(IV)} - S_{FL}v^{(IV)} &= q_z \\ S_{EL}u''' + S_{LT}\varphi''' - S_{FL}w^{(IV)} + EI_z v^{(IV)} &= q_y. \end{aligned} \tag{2.15}$$

At $x = 0$ and $x = \ell$, boundary conditions can be expressed as follows:

$$\begin{aligned}
&u = 0 && \text{or} && EAu' + S_{ET}\varphi' - S_{EF}w'' + S_{EL}v'' = \hat{f}_x \\
&\varphi = 0 && \text{or} && S_{ET}u' + GJ\varphi' - S_{FT}w'' + S_{LT}v'' = \hat{m}_x \\
&w' = 0 && \text{or} && -S_{EF}u' - S_{FT}\varphi' + EI_y w'' - S_{FL}v'' = -\hat{m}_y \\
&v' = 0 && \text{or} && S_{EL}u' + S_{LT}\varphi' - S_{FL}w'' + EI_z v'' = \hat{m}_z \\
&w = 0 && \text{or} && S_{EF}u'' + S_{FT}\varphi'' - EI_y w''' + S_{FL}v''' = \hat{f}_z \\
&v = 0 && \text{or} && -S_{EL}u'' - S_{LT}\varphi'' + S_{FL}w''' - EI_z v''' = \hat{f}_y,
\end{aligned} \tag{2.16}$$

where $\hat{f}_x$, $\hat{f}_y$, $\hat{f}_z$ are the tip loads in the x, y and z direction respectively, $\hat{m}_x$ is the tip torque, and $\hat{m}_y$, $\hat{m}_z$ are the tip moments about the y and z axes respectively.

Let $\hat{\boldsymbol{K}}$ be a block matrix such that:

$$\hat{\boldsymbol{K}} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B}^T & \boldsymbol{D} \end{bmatrix}, \tag{2.17}$$

where

$$\boldsymbol{A} = \begin{bmatrix} EA & S_{ET} \\ S_{ET} & GJ \end{bmatrix}, \tag{2.18a}$$

$$\boldsymbol{B} = \begin{bmatrix} S_{EF} & -S_{EL} \\ S_{FT} & -S_{LT} \end{bmatrix}, \tag{2.18b}$$

$$\boldsymbol{D} = \begin{bmatrix} EI_y & -S_{FL} \\ -S_{FL} & EI_z \end{bmatrix}. \tag{2.18c}$$

Now consider the matrix

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K_A} & \boldsymbol{K_B} \\ \boldsymbol{K_B}^T & \boldsymbol{K_D} \end{bmatrix}, \tag{2.19}$$

such that $\boldsymbol{K} = \hat{\boldsymbol{K}}^{-1}$, i.e. $\boldsymbol{K}$ is an inverse of matrix $\hat{\boldsymbol{K}}$. Entries of $\boldsymbol{K}$ can be determined using the blockwise inversion formula [33]:

$$\boldsymbol{K_A} = \boldsymbol{A}^{-1} + \boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{D} - \boldsymbol{B}^T\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{B}^T\boldsymbol{A}^{-1}, \tag{2.20a}$$

$$\boldsymbol{K_B} = -\boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{D} - \boldsymbol{B}^T\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}, \tag{2.20b}$$

$$\boldsymbol{K_B}^T = -(\boldsymbol{D} - \boldsymbol{B}^T\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{B}^T\boldsymbol{A}^{-1}, \tag{2.20c}$$

$$\boldsymbol{K_D} = (\boldsymbol{D} - \boldsymbol{B}^T\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}. \tag{2.20d}$$

Using matrix (2.17), the governing Equations (2.15) and boundary conditions (2.16) can be written in a compact matrix form as follows:

$$-\boldsymbol{A}\begin{bmatrix} u \\ \varphi \end{bmatrix}'' + \boldsymbol{B}\begin{bmatrix} w \\ v \end{bmatrix}''' = \begin{bmatrix} q_x \\ q_\varphi \end{bmatrix}, \tag{2.21a}$$

$$-\boldsymbol{B}^T\begin{bmatrix} u \\ \varphi \end{bmatrix}''' + \boldsymbol{D}\begin{bmatrix} w \\ v \end{bmatrix}^{(IV)} = \begin{bmatrix} q_z \\ q_y \end{bmatrix}. \tag{2.21b}$$

$$\begin{bmatrix} u \\ \varphi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{2.22a}$$

$$\begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{2.22b}$$

$$\begin{bmatrix} w \\ v \end{bmatrix}' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.22c}$$

$$\boldsymbol{A}\begin{bmatrix} u \\ \varphi \end{bmatrix}' - \boldsymbol{B}\begin{bmatrix} w \\ v \end{bmatrix}'' = \begin{bmatrix} \hat{f}_x \\ \hat{m}_x \end{bmatrix}, \tag{2.23a}$$

$$-\boldsymbol{B}^T\begin{bmatrix} u \\ \varphi \end{bmatrix}' + \boldsymbol{D}\begin{bmatrix} w \\ v \end{bmatrix}'' = \begin{bmatrix} -\hat{m}_y \\ \hat{m}_z \end{bmatrix}, \tag{2.23b}$$

$$\boldsymbol{B}^T\begin{bmatrix} u \\ \varphi \end{bmatrix}'' - \boldsymbol{D}\begin{bmatrix} w \\ v \end{bmatrix}''' = \begin{bmatrix} \hat{f}_z \\ \hat{f}_y \end{bmatrix}. \tag{2.23c}$$

The procedure of obtaining the exact analytical solution of the system of Eqs. (2.21) is described in the following section.

3. Direct exact solution

To derive the exact solution of Eqs. (2.21), Eq. (2.21a) is differentiated and rearranged. As opposed to Doeva et al. [20], it is assumed that the loads at the right hand side of Eqs. (2.21) are uniformly distributed, thus the third derivative of the vector $[u \quad \varphi]^T$ can be obtained as follows:

$$\begin{bmatrix} u \\ \varphi \end{bmatrix}''' = \boldsymbol{A}^{-1}\boldsymbol{B}\begin{bmatrix} w \\ v \end{bmatrix}^{(IV)}, \tag{3.1}$$

noting that $\boldsymbol{A}^{-1}\boldsymbol{B}$ can be thought of as an eccentricity matrix (with dimensions of length) where it represents the eccentricity of the neutral axis from the midplane as well as the shift in shear centre from the centre of the beam. It is noted here that for the case of non-uniform loads as reported in [20], the first derivative of non-uniform loads does not disappear in the expression of the third derivative of U. This results in different expressions for the exact solutions as shown in the following analysis. Substituting Eq. (3.1) into Eq. (2.21b), rearranging it and using Eqs. (2.20), the expression for the fourth derivative of the vector $[w \quad v]^T$ can be obtained:

$$\begin{bmatrix} w \\ v \end{bmatrix}^{IV} = \boldsymbol{K_D}\begin{bmatrix} q_z \\ q_y \end{bmatrix}. \tag{3.2}$$

Integrating Eq. (3.2), expressions for the third, second and first derivatives of $[w \quad v]^T$ are derived:

$$\begin{bmatrix} w \\ v \end{bmatrix}''' = \boldsymbol{K_D}\begin{bmatrix} q_z \\ q_y \end{bmatrix}x + \boldsymbol{C}_1, \tag{3.3}$$

$$\begin{bmatrix} w \\ v \end{bmatrix}'' = \frac{1}{2}\boldsymbol{K_D}\begin{bmatrix} q_z \\ q_y \end{bmatrix}x^2 + \boldsymbol{C}_1 x + \boldsymbol{C}_2, \tag{3.4}$$

$$\begin{bmatrix} w \\ v \end{bmatrix}' = \frac{1}{6}\boldsymbol{K_D}\begin{bmatrix} q_z \\ q_y \end{bmatrix}x^3 + \frac{1}{2}\boldsymbol{C}_1 x^2 + \boldsymbol{C}_2 x + \boldsymbol{C}_3. \tag{3.5}$$

Integrating Eq. (3.5) again, the exact solution for the out-of-plane displacement w and in-plane displacement v can be obtained:

$$\begin{bmatrix} w \\ v \end{bmatrix} = \frac{1}{24}\boldsymbol{K_D}\begin{bmatrix} q_z \\ q_y \end{bmatrix}x^4 + \frac{1}{6}\boldsymbol{C}_1 x^3 + \frac{1}{2}\boldsymbol{C}_2 x^2 + \boldsymbol{C}_3 x + \boldsymbol{C}_4, \tag{3.6}$$

where $\boldsymbol{C}_i$, $i = 1, 2, \ldots, 4$, are the vectors of unknown integrating constants to be determined.

Substituting Eq. (3.3) into Eq. (2.21a) and rearranging it, the following expression is obtained:

$$\begin{bmatrix} u \\ \varphi \end{bmatrix}'' = -\boldsymbol{K_B}\begin{bmatrix} q_z \\ q_y \end{bmatrix}x + \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{C}_1 - \boldsymbol{A}^{-1}\begin{bmatrix} q_x \\ q_\varphi \end{bmatrix}. \tag{3.7}$$

Integrating Eq. (3.7) twice, the first derivative of the vector $[u \quad \varphi]^T$ and the exact solution for the displacement u and twist φ can be derived:

$$\begin{bmatrix} u \\ \varphi \end{bmatrix}' = -\frac{1}{2}\boldsymbol{K_B}\begin{bmatrix} q_z \\ q_y \end{bmatrix}x^2 + \quad \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{C}_1 x - \boldsymbol{A}^{-1}\begin{bmatrix} q_x \\ q_\varphi \end{bmatrix}x + \boldsymbol{C}_5, \tag{3.8}$$

$$\begin{bmatrix} u \\ \varphi \end{bmatrix} = -\frac{1}{6}\boldsymbol{K_B}\begin{bmatrix} q_z \\ q_y \end{bmatrix}x^3 + \frac{1}{2}\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{C}_1 x^2 - \frac{1}{2}\boldsymbol{A}^{-1}\begin{bmatrix} q_x \\ q_\varphi \end{bmatrix}x^2 + \boldsymbol{C}_5 x + \boldsymbol{C}_6, \tag{3.9}$$

where $\boldsymbol{C}_i$, $i = 1, 5, 6$, are the vectors of unknown coefficients to be determined.

Eqs. (3.9) and (3.6) are the general exact solutions for the 3D static deflection of a fully coupled composite beam under the action of uniformly distributed and tip loads, unrestricted by the type of boundary conditions. These general solutions have six vectors of unknown constants to be derived. Once the boundary conditions are given, these unknown vectors can be obtained and the exact particular solutions can be found. In comparison with the case of non-uniformly distributed loads [20], then the solutions presented in Eqs. (3.6) and (3.9) cannot be retrieved from those of Doeva et al. [20], due to the different expressions for the third derivative of $[u \quad \varphi]^T$.

In the following subsections the expressions of the unknowns for various boundary conditions for tip or uniformly distributed loads are provided. Substituting these expressions into Eqs. (3.9) and (3.6), the exact solutions for every particular case can be obtained.

3.1. Cantilever composite beam under tip loads

Consider a cantilever composite beam under tip loads applied at the free end. To obtain the expressions for the vectors $\boldsymbol{C}_i$, $i = 1, 2, \dots, 6$, for this case, boundary conditions (2.22) should be applied at $x = 0$ and boundary conditions (2.23) at $x = \ell$. In addition, q_u, q_φ, q_z and q_y are set to be zero. Thus, using Eqs. (2.20), the vectors $\boldsymbol{C}_i$, $i = 1, 2, \dots, 6$, have the following form:

$$\boldsymbol{C}_1 = -\boldsymbol{K_D}\begin{bmatrix}\hat{f}_z\\ \hat{f}_y\end{bmatrix}, \tag{3.10}$$

$$\boldsymbol{C}_2 = \boldsymbol{K_D}\begin{bmatrix}-\hat{m}_y\\ \hat{m}_z\end{bmatrix} + \boldsymbol{K_D}\begin{bmatrix}\hat{f}_z\\ \hat{f}_y\end{bmatrix}\ell - \boldsymbol{K}_{\boldsymbol{B}}^T\begin{bmatrix}\hat{f}_x\\ \hat{m}_x\end{bmatrix}, \tag{3.11}$$

$$\boldsymbol{C}_5 = -\boldsymbol{K_B}\begin{bmatrix}-\hat{m}_y\\ \hat{m}_z\end{bmatrix} - \boldsymbol{K_B}\begin{bmatrix}\hat{f}_z\\ \hat{f}_y\end{bmatrix}\ell + \boldsymbol{K_A}\begin{bmatrix}\hat{f}_x\\ \hat{m}_x\end{bmatrix}, \tag{3.12}$$

$$\boldsymbol{C}_3 = \boldsymbol{C}_4 = \boldsymbol{C}_6 = [0 \quad 0]^T. \tag{3.13}$$

3.2. Cantilever composite beam under distributed loads

Consider a cantilever composite beam subject to uniformly distributed loads. Applying boundary conditions (2.22) at the clamped end and boundary conditions (2.23) at the free end with

$$\begin{bmatrix}\hat{f}_x\\ \hat{m}_x\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}, \qquad \begin{bmatrix}-\hat{m}_y\\ \hat{m}_z\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}, \qquad \begin{bmatrix}\hat{f}_z\\ \hat{f}_y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix},$$

and using Eqs. (2.20), the following expressions for the unknown vectors $\boldsymbol{C}_i$, $i = 1, 2 \dots, 6$, can be found:

$$\boldsymbol{C}_1 = \boldsymbol{K}_{\boldsymbol{B}}^T\begin{bmatrix}q_x\\ q_\varphi\end{bmatrix} - \boldsymbol{K_D}\begin{bmatrix}q_z\\ q_y\end{bmatrix}\ell, \tag{3.14}$$

$$\boldsymbol{C}_2 = -\boldsymbol{K}_{\boldsymbol{B}}^T\begin{bmatrix}q_x\\ q_\varphi\end{bmatrix}\ell + \frac{1}{2}\boldsymbol{K_D}\begin{bmatrix}q_z\\ q_y\end{bmatrix}\ell^2, \tag{3.15}$$

$$\boldsymbol{C}_5 = \boldsymbol{K_A}\begin{bmatrix}q_x\\ q_\varphi\end{bmatrix}\ell - \frac{1}{2}\boldsymbol{K_B}\begin{bmatrix}q_z\\ q_y\end{bmatrix}\ell^2, \tag{3.16}$$

$$\boldsymbol{C}_3 = \boldsymbol{C}_4 = \boldsymbol{C}_6 = [0 \quad 0]^T. \tag{3.17}$$

3.3. Composite beam clamped at both ends subject to distributed loads

Applying boundary conditions (2.22) at both ends of a fully clamped composite beam along with

$$\begin{bmatrix}\hat{f}_x\\ \hat{m}_x\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}, \qquad \begin{bmatrix}-\hat{m}_y\\ \hat{m}_z\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}, \qquad \begin{bmatrix}\hat{f}_z\\ \hat{f}_y\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix},$$

and using Eqs. (2.20), the following expressions for the unknown vectors $\boldsymbol{C}_i$, $i = 1, 2 \dots, 6$, can be determined:

$$\boldsymbol{C}_1 = -\frac{1}{2}\boldsymbol{K_D}\begin{bmatrix}q_z\\ q_y\end{bmatrix}\ell, \tag{3.18}$$

$$\boldsymbol{C}_2 = \frac{1}{12}\boldsymbol{K_D}\begin{bmatrix}q_z\\ q_y\end{bmatrix}\ell^2, \tag{3.19}$$

$$\boldsymbol{C}_5 = \frac{1}{2}\boldsymbol{A}^{-1}\begin{bmatrix}q_x\\ q_\varphi\end{bmatrix}\ell - \frac{1}{12}\boldsymbol{K_B}\begin{bmatrix}q_z\\ q_y\end{bmatrix}\ell^2, \tag{3.20}$$

$$\boldsymbol{C}_3 = \boldsymbol{C}_4 = \boldsymbol{C}_6 = [0 \quad 0]^T. \tag{3.21}$$

3.4. Composite beam clamped at one end and simply supported at another subject to distributed loads

For a composite beam, clamped at one end and simply supported at the other end, boundary conditions (2.22) should be applied at $x = 0$ and boundary conditions (2.22a), (2.22b), (2.23b) at $x = \ell$. As a result, using Eqs. (2.20), the following expressions for the unknown vectors $\boldsymbol{C}_i$, $i = 1, 2, \dots, 6$, can be obtained:

$$\boldsymbol{C}_1 = -\frac{1}{2}\boldsymbol{P}\boldsymbol{B}^T\boldsymbol{A}^{-1}\begin{bmatrix}q_x\\ q_\varphi\end{bmatrix} - \left(\frac{1}{2}\boldsymbol{P} - \boldsymbol{P}\boldsymbol{Q}\boldsymbol{K_D}\right)\begin{bmatrix}q_z\\ q_y\end{bmatrix}\ell, \tag{3.22}$$

$$\boldsymbol{C}_2 = \frac{1}{6}\boldsymbol{P}\boldsymbol{B}^T\boldsymbol{A}^{-1}\begin{bmatrix}q_x\\ q_\varphi\end{bmatrix}\ell + \left(\frac{1}{6}\boldsymbol{P} - \frac{1}{12}\boldsymbol{K_D} - \frac{1}{3}\boldsymbol{P}\boldsymbol{Q}\boldsymbol{K_D}\right)\begin{bmatrix}q_z\\ q_y\end{bmatrix}\ell^2, \tag{3.23}$$

$$\begin{aligned}\boldsymbol{C}_5 = &\left(\frac{1}{2}\boldsymbol{A}^{-1} + \frac{1}{4}\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{P}\boldsymbol{B}^T\boldsymbol{A}^{-1}\right)\begin{bmatrix}q_x\\ q_\varphi\end{bmatrix}\ell\\ &+\left(\frac{1}{6}\boldsymbol{K_B} + \frac{1}{4}\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{P} - \frac{1}{2}\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{P}\boldsymbol{Q}\boldsymbol{K_D}\right)\begin{bmatrix}q_z\\ q_y\end{bmatrix}\ell^2,\end{aligned} \tag{3.24}$$

$$\boldsymbol{C}_3 = \boldsymbol{C}_4 = \boldsymbol{C}_6 = [0 \quad 0]^T, \tag{3.25}$$

where $\boldsymbol{I}$ is the identity matrix and

$$\boldsymbol{P} = \left(\frac{2}{3}\boldsymbol{D} - \frac{1}{2}\boldsymbol{B}^T\boldsymbol{A}^{-1}\boldsymbol{B}\right)^{-1},$$
$$\boldsymbol{Q} = \frac{1}{12}\boldsymbol{D} - \frac{1}{6}\boldsymbol{B}^T\boldsymbol{A}^{-1}\boldsymbol{B}.$$

3.5. Composite beam simply supported at both ends subject to distributed loads

For a composite beam simply supported at both ends boundary conditions (2.22a), (2.22b) and (2.23b) should be applied at both $x = 0$ and $x = \ell$. Using Eqs. (2.20), the following expressions for the unknown vectors $\boldsymbol{C}_i$, $i = 1, 2, \dots, 6$, can be derived:

$$\boldsymbol{C}_1 = \boldsymbol{K_B}^T\begin{bmatrix}q_x\\ q_\varphi\end{bmatrix} - \frac{1}{2}\boldsymbol{K_D}\begin{bmatrix}q_z\\ q_y\end{bmatrix}\ell, \tag{3.26}$$

$$\boldsymbol{C}_2 = \frac{1}{2}\boldsymbol{D}^{-1}\boldsymbol{B}^T\boldsymbol{K_A}\begin{bmatrix}q_x\\ q_\varphi\end{bmatrix}\ell - \frac{1}{12}\boldsymbol{D}^{-1}\boldsymbol{B}^T\boldsymbol{K_B}\begin{bmatrix}q_z\\ q_y\end{bmatrix}\ell^2, \tag{3.27}$$

$$\boldsymbol{C}_3 = -\left(\frac{1}{6}\boldsymbol{K_B}^T + \frac{1}{4}\boldsymbol{D}^{-1}\boldsymbol{B}^T\boldsymbol{K_A}\right)\begin{bmatrix}q_x\\ q_\varphi\end{bmatrix}\ell^2 + \frac{1}{24}\left(\boldsymbol{K_D} + \boldsymbol{D}^{-1}\boldsymbol{B}^T\boldsymbol{K_B}\right)\begin{bmatrix}q_z\\ q_y\end{bmatrix}\ell^3, \tag{3.28}$$

$$\boldsymbol{C}_5 = \frac{1}{2}\boldsymbol{K_A}\begin{bmatrix}q_x\\ q_\varphi\end{bmatrix}\ell - \frac{1}{12}\boldsymbol{K_B}\begin{bmatrix}q_z\\ q_y\end{bmatrix}\ell^2, \tag{3.29}$$

$$\boldsymbol{C}_4 = \boldsymbol{C}_6 = [0 \quad 0]^T. \tag{3.30}$$

4. Comparison with the closed-form solution from Timoshenko beam theory

Doeva et al. [21] presented a closed-form solution for the linear static deflection of fully coupled Timoshenko composite beams. Neglecting transverse shear terms, it is possible to obtain the exact solution for Euler–Bernoulli composite beams which is proven to be identical to the solution obtained directly from the governing equations.

The solution provided by Doeva et al. [21] is:

$$\boldsymbol{\Theta} = \frac{1}{6}x^3\boldsymbol{T}\boldsymbol{e}\boldsymbol{f} - \frac{1}{2}x^2(\boldsymbol{Z}^T\boldsymbol{f} + \boldsymbol{T}(\boldsymbol{e}\overline{\boldsymbol{C}}_1 + \boldsymbol{m})) + x(\boldsymbol{Z}^T\overline{\boldsymbol{C}}_1 + \boldsymbol{T}\overline{\boldsymbol{C}}_2) + \overline{\boldsymbol{C}}_3. \tag{4.1}$$

$$\begin{aligned}\boldsymbol{V} = &-\frac{1}{24}x^4\boldsymbol{e}\boldsymbol{T}\boldsymbol{e}\boldsymbol{f} + \frac{1}{6}x^3(\boldsymbol{Z}\boldsymbol{e}\boldsymbol{f} + \boldsymbol{e}\boldsymbol{Z}^T\boldsymbol{f} + \boldsymbol{e}\boldsymbol{T}(\boldsymbol{e}\overline{\boldsymbol{C}}_1 + \boldsymbol{m}))\\ &-\frac{1}{2}x^2(\boldsymbol{R}\boldsymbol{f} + \boldsymbol{Z}(\boldsymbol{e}\overline{\boldsymbol{C}}_1 + \boldsymbol{m}) + \boldsymbol{e}\boldsymbol{Z}^T\overline{\boldsymbol{C}}_1 + \boldsymbol{e}\boldsymbol{T}\overline{\boldsymbol{C}}_2)\\ &+x(\boldsymbol{R}\overline{\boldsymbol{C}}_1 + \boldsymbol{Z}\overline{\boldsymbol{C}}_2 - \boldsymbol{e}\overline{\boldsymbol{C}}_3) + \overline{\boldsymbol{C}}_4,\end{aligned} \tag{4.2}$$

where the rotation and displacement column matrices $\boldsymbol{\Theta}$ and $\boldsymbol{V}$, the vectors of forces and moments $\boldsymbol{F}$ and $\boldsymbol{M}$, matrix $\boldsymbol{e}$ and vector of constants $\overline{\boldsymbol{C}}_i$, $i = 1,2,3,4$ have the following form:

$$\boldsymbol{\Theta} = \begin{bmatrix} \varphi \\ \theta_y \\ \theta_z \end{bmatrix}, \quad \boldsymbol{V} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \quad \boldsymbol{f} = \begin{bmatrix} q_x \\ q_y \\ q_z \end{bmatrix}, \quad \boldsymbol{m} = \begin{bmatrix} q_\phi \\ m_y \\ m_z \end{bmatrix},$$

$$\boldsymbol{e} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\overline{\boldsymbol{C}}_1 = \begin{bmatrix} C_{1,1} \\ C_{1,2} \\ C_{1,3} \end{bmatrix}, \quad \overline{\boldsymbol{C}}_2 = \begin{bmatrix} C_{2,1} \\ C_{2,2} \\ C_{2,3} \end{bmatrix}, \quad \overline{\boldsymbol{C}}_3 = \begin{bmatrix} C_{3,1} \\ C_{3,2} \\ C_{3,3} \end{bmatrix}, \quad \overline{\boldsymbol{C}}_4 = \begin{bmatrix} C_{4,1} \\ C_{4,2} \\ C_{4,3} \end{bmatrix},$$

where θ_y and θ_z are rotations about y and z axes respectively. In this formulation a 6×6 compliance block matrix $\hat{\Sigma}$ is used:

$$\hat{\boldsymbol{\Sigma}} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{Z} \\ \boldsymbol{Z}^T & \boldsymbol{T} \end{bmatrix}, \tag{4.3}$$

where $\boldsymbol{R}$, $\boldsymbol{Z}$, $\boldsymbol{Z}^T$ and $\boldsymbol{T}$ are 3×3 matrices. In the current paper the Euler–Bernoulli theory is adopted, so in order for both formulations to be consistent, the effects of transverse shear deformations are discarded and the compliance matrix (4.3) is expressed as follows:

$$\hat{\boldsymbol{\Sigma}}_{6\times6} = \begin{bmatrix} R_{11} & 0 & 0 & Z_{11} & Z_{12} & Z_{13} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ Z_{11} & 0 & 0 & T_{11} & T_{12} & T_{13} \\ Z_{12} & 0 & 0 & T_{12} & T_{22} & T_{23} \\ Z_{13} & 0 & 0 & T_{13} & T_{23} & T_{33} \end{bmatrix}. \tag{4.4}$$

Now using matrix (4.4), Eqs. (4.1) and (4.2) can be rewritten in matrix form:

$$\begin{aligned} \begin{bmatrix} w \\ v \end{bmatrix} = & \frac{1}{24}x^4 \begin{bmatrix} T_{22} & -T_{23} \\ -T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} q_z \\ q_y \end{bmatrix} + \frac{1}{6}x^3 \begin{bmatrix} Z_{12} & T_{12} \\ -Z_{13} & -T_{13} \end{bmatrix} \begin{bmatrix} q_x \\ q_\phi \end{bmatrix} \\ & -\frac{1}{6}x^3 \begin{bmatrix} T_{22} & -T_{23} \\ -T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} C_{1,3} \\ C_{1,2} \end{bmatrix} \\ & -\frac{1}{2}x^2 \begin{bmatrix} T_{22} & -T_{23} \\ -T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} C_{2,2} \\ -C_{2,3} \end{bmatrix} - \frac{1}{2}x^2 \begin{bmatrix} Z_{12} & T_{12} \\ -Z_{13} & -T_{13} \end{bmatrix} \begin{bmatrix} C_{1,1} \\ C_{2,1} \end{bmatrix} \\ & +x \begin{bmatrix} -C_{3,2} \\ C_{3,3} \end{bmatrix} + \begin{bmatrix} C_{4,3} \\ C_{4,2} \end{bmatrix}. \end{aligned} \tag{4.5}$$

$$\begin{aligned} \begin{bmatrix} u \\ \varphi \end{bmatrix} = & -\frac{1}{6}x^3 \begin{bmatrix} Z_{12} & -Z_{13} \\ T_{12} & -T_{13} \end{bmatrix} \begin{bmatrix} q_z \\ q_y \end{bmatrix} - \frac{1}{2}x^2 \begin{bmatrix} R_{11} & Z_{11} \\ Z_{11} & T_{11} \end{bmatrix} \begin{bmatrix} q_x \\ q_\phi \end{bmatrix} \\ & +\frac{1}{2}x^2 \begin{bmatrix} Z_{12} & -Z_{13} \\ T_{12} & -T_{13} \end{bmatrix} \begin{bmatrix} C_{1,3} \\ C_{1,2} \end{bmatrix} \\ & +x \begin{bmatrix} Z_{12} & -Z_{13} \\ T_{12} & -T_{13} \end{bmatrix} \begin{bmatrix} C_{2,2} \\ -C_{2,3} \end{bmatrix} + x \begin{bmatrix} R_{11} & Z_{11} \\ Z_{11} & T_{11} \end{bmatrix} \begin{bmatrix} C_{1,1} \\ C_{2,1} \end{bmatrix} + \begin{bmatrix} C_{4,1} \\ C_{3,1} \end{bmatrix}, \end{aligned} \tag{4.6}$$

Now we define a 4×4 block matrix $\boldsymbol{\Sigma}$ as:

$$\boldsymbol{\Sigma} = \begin{bmatrix} \boldsymbol{\Sigma}_A & \boldsymbol{\Sigma}_B \\ \boldsymbol{\Sigma}_B^T & \boldsymbol{\Sigma}_D \end{bmatrix}, \tag{4.7}$$

where

$$\boldsymbol{\Sigma}_A = \begin{bmatrix} R_{11} & Z_{11} \\ Z_{11} & T_{11} \end{bmatrix}, \tag{4.8a}$$

$$\boldsymbol{\Sigma}_B = \begin{bmatrix} Z_{12} & -Z_{13} \\ T_{12} & -T_{13} \end{bmatrix}, \tag{4.8b}$$

$$\boldsymbol{\Sigma}_B^T = \begin{bmatrix} Z_{21} & T_{21} \\ -Z_{13} & -T_{13} \end{bmatrix}, \tag{4.8c}$$

$$\boldsymbol{\Sigma}_D = \begin{bmatrix} T_{22} & -T_{23} \\ -T_{23} & T_{33} \end{bmatrix}. \tag{4.8d}$$

Using the definitions in (4.8), Eqs. (4.5) and (4.6) can be rewritten as:

$$\begin{bmatrix} w \\ v \end{bmatrix} = \frac{1}{24}\boldsymbol{\Sigma}_D \begin{bmatrix} q_z \\ q_y \end{bmatrix} x^4 + \frac{1}{6}\left(\boldsymbol{\Sigma}_B^T \begin{bmatrix} q_x \\ q_\phi \end{bmatrix} - \boldsymbol{\Sigma}_D \boldsymbol{C}_1\right) x^3 + \frac{1}{2}\boldsymbol{C}_2 x^2 + \boldsymbol{C}_3 x + \boldsymbol{C}_4. \tag{4.9}$$

$$\begin{bmatrix} u \\ \varphi \end{bmatrix} = -\frac{1}{6}\boldsymbol{\Sigma}_B \begin{bmatrix} q_z \\ q_y \end{bmatrix} x^3 + \frac{1}{2}\left(\boldsymbol{\Sigma}_B \boldsymbol{C}_1 - \boldsymbol{\Sigma}_A \begin{bmatrix} q_x \\ q_\phi \end{bmatrix}\right) x^2 + \boldsymbol{C}_5 x + \boldsymbol{C}_6, \tag{4.10}$$

where,

$$\boldsymbol{C}_1 = \begin{bmatrix} C_{1,3} \\ C_{1,2} \end{bmatrix}, \quad \boldsymbol{C}_2 = \boldsymbol{\Sigma}_D \begin{bmatrix} -C_{2,2} \\ C_{2,3} \end{bmatrix} + \boldsymbol{\Sigma}_B^T \begin{bmatrix} -C_{1,1} \\ -C_{2,1} \end{bmatrix}, \quad \boldsymbol{C}_3 = \begin{bmatrix} -C_{3,2} \\ C_{3,3} \end{bmatrix},$$

$$\boldsymbol{C}_4 = \begin{bmatrix} C_{4,3} \\ C_{4,2} \end{bmatrix},$$

$$\boldsymbol{C}_5 = \boldsymbol{\Sigma}_B \begin{bmatrix} C_{2,2} \\ -C_{2,3} \end{bmatrix} + \boldsymbol{\Sigma}_A \begin{bmatrix} C_{1,1} \\ C_{2,1} \end{bmatrix}, \quad \boldsymbol{C}_6 = \begin{bmatrix} C_{4,1} \\ C_{3,1} \end{bmatrix}.$$

In order to demonstrate that Eqs. (4.9) and (4.10) are identical to Eqs. (3.6) and (3.9) respectively, we rewrite (3.6) and (3.9) as

$$\begin{bmatrix} w \\ v \end{bmatrix} = \frac{1}{24}\boldsymbol{K}_D \begin{bmatrix} q_z \\ q_y \end{bmatrix} x^4 + \frac{1}{6}\left(\boldsymbol{K}_B^T \begin{bmatrix} q_x \\ q_\phi \end{bmatrix} - \boldsymbol{K}_D \hat{\boldsymbol{C}}_1\right) x^3 + \frac{1}{2}\boldsymbol{C}_2 x^2 + \boldsymbol{C}_3 x + \boldsymbol{C}_4, \tag{4.11}$$

$$\begin{bmatrix} u \\ \varphi \end{bmatrix} = -\frac{1}{6}\boldsymbol{K}_B \begin{bmatrix} q_z \\ q_y \end{bmatrix} x^3 + \frac{1}{2}\left(\boldsymbol{K}_B \hat{\boldsymbol{C}}_1 - \boldsymbol{K}_A \begin{bmatrix} q_x \\ q_\varphi \end{bmatrix}\right) x^2 + \boldsymbol{C}_5 x + \boldsymbol{C}_6, \tag{4.12}$$

where,

$$\hat{\boldsymbol{C}}_1 = \boldsymbol{K}_B^{-1}\left(\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{C}_1 + \left(\boldsymbol{K}_A - \boldsymbol{A}^{-1}\right)\begin{bmatrix} q_x \\ q_\varphi \end{bmatrix}\right),$$

then we prove that $\boldsymbol{K} = \boldsymbol{\Sigma}$. Towards this goal, the following proposition is introduced:

Proposition 4.1. *Let $\mathcal{A}$ be an $n \times n$ matrix:*

$$\mathcal{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n-1} & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n-1} & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-11} & a_{n-12} & \cdots & a_{n-1n-1} & a_{n-1n} \\ a_{n1} & a_{n2} & \cdots & a_{nn-1} & a_{nn} \end{bmatrix}.$$

Consider an operator $()^+$ such that

$$\begin{aligned} a_{in}^+ &= -a_{in}, \\ a_{nj}^+ &= -a_{nj}, \quad i,j = 1,2,\ldots,n-1, \end{aligned} \tag{4.13}$$

i.e., matrix $\mathcal{A}^+$ has the following representation:

$$\mathcal{A}^+ = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n-1} & -a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n-1} & -a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n-11} & a_{n-12} & \cdots & a_{n-1n-1} & -a_{n-1n} \\ -a_{n1} & -a_{n2} & \cdots & -a_{nn-1} & a_{nn} \end{bmatrix}.$$

Then for any $\mathcal{A}$ and $\mathcal{B}$ $n \times n$ matrices, the following holds:

$$(\mathcal{A}\mathcal{B})^+ = \mathcal{A}^+\mathcal{B}^+. \tag{4.14}$$

For a proof of Eq. (4.14) see Appendix A.

Consider a 4×4 matrix $\hat{\boldsymbol{\Sigma}}_{4\times4}$ which is obtained by discarding the second and third rows and columns from matrix (4.4):

$$\hat{\boldsymbol{\Sigma}}_{4\times4} = \begin{bmatrix} R_{11} & Z_{11} & Z_{12} & Z_{13} \\ Z_{11} & T_{11} & T_{12} & T_{13} \\ Z_{12} & T_{21} & T_{22} & T_{23} \\ Z_{13} & T_{31} & T_{32} & T_{33} \end{bmatrix}, \tag{4.15}$$

where $\hat{\boldsymbol{\Sigma}}_{4\times4}$ is the compliance matrix of the beam model and obviously, $\hat{\boldsymbol{\Sigma}}_{4\times4} = \boldsymbol{S}^{-1}$. Considering the characteristics of the operator $()^+$ defined in Eqs. (4.13), the following relations hold:

$$\hat{\boldsymbol{K}} = \boldsymbol{S}^+, \tag{4.16a}$$

$$\boldsymbol{\Sigma} = \hat{\boldsymbol{\Sigma}}^+_{4\times 4}. \tag{4.16b}$$

Taking the inverse of both sides of Eq. (4.16a), we obtain:

$$\hat{\boldsymbol{K}}^{-1} = \left(\boldsymbol{S}^+\right)^{-1} = \boldsymbol{K}. \tag{4.17}$$

From the Eq. (4.16b) it is also clear that

$$\hat{\boldsymbol{\Sigma}}^+_{4\times 4} = \left(\boldsymbol{S}^{-1}\right)^+ = \boldsymbol{\Sigma}. \tag{4.18}$$

Since stiffness matrix $\boldsymbol{S}$ is positive definite, the following holds:

$$\boldsymbol{S}^{-1}\boldsymbol{S} = \boldsymbol{I}, \tag{4.19a}$$

$$(\boldsymbol{S}^{-1}\boldsymbol{S})^+ = \boldsymbol{I}^+. \tag{4.19b}$$

By definition (4.13), $\boldsymbol{I}^+ = \boldsymbol{I}$. Using Eq. (4.14), Eq. (4.19b) is rewritten as:

$$(\boldsymbol{S}^{-1})^+\boldsymbol{S}^+ = \boldsymbol{I},$$
$$\boldsymbol{S}^+ = ((\boldsymbol{S}^{-1})^+)^{-1},$$
$$(\boldsymbol{S}^+)^{-1} = (((\boldsymbol{S}^{-1})^+)^{-1})^{-1},$$
$$(\boldsymbol{S}^+)^{-1} = (\boldsymbol{S}^{-1})^+.$$

As $(\boldsymbol{S}^+)^{-1} = (\boldsymbol{S}^{-1})^+$, from the Eqs. (4.17) and (4.18), it follows that $\boldsymbol{K} = \boldsymbol{\Sigma}$. Thus, it is clear that Eqs. (4.9) and (4.10) are identical to Eqs. (4.11) and (4.12).

5. Chebyshev collocation method

To obtain a numerical solution for Eq. (2.15), the computationally efficient and accurate Chebyshev collocation method (CCM) is applied.

Chebyshev polynomials $T_i(x)$ are used as trial functions to discretise unknown variables u, v, w and φ while the Chebyshev points are employed as collocation points (for more details of CCM see [34,35])

$$u = \sum_{i=0}^{N} a_i T_i(x),\ \ \varphi = \sum_{i=0}^{N} b_i T_i(x),\ \ w = \sum_{i=0}^{N} c_i T_i(x),\ \ v = \sum_{i=0}^{N} d_i T_i(x). \tag{5.1}$$

Eqs. (5.1) are substituted into the governing Equations (2.15) and the residuals at the collocation points set to zero. To have a well-posed system of $4\times(N+1)$ equations in conjunction with the boundary conditions (2.16), in the case of u and φ, the equations associated with the residuals at the first and the last Chebyshev points are eliminated, whereas in the case of w and v, those equations associated with the first two and the last two Chebyshev points are discarded systematically. Finally, this system of linear equations is solved for unknown coefficients a_i, b_i, c_i and d_i where $i = 0, 1, 2, \dots, N$. It is noted that $N = 5$ was sufficient to obtain converged results.

6. Verification studies

In order to verify results obtained from the proposed approach, a number of test cases of laminated composite beams under uniformly distributed load q_z are considered. Normalised deflection $\overline{w} = \dfrac{100wbh^3E_{22}}{q_z\ell^4}$ (where b is the width and h is the thickness of rectangular beam cross-section) is obtained for slender composite beams ($\ell/h = 50$) with symmetric [0°/90°/0°] and anti-symmetric [0°/90°] lay-ups for different boundary conditions "H–H: Hinged–Hinged", "C–H: Clamped–Hinged", "C–C: Clamped–Clamped", "C–F: Clamped–Free" and compared against some available results in the literature. The material properties used here are listed as follows:

$$E_{11}/E_{22} = 25, \quad G_{12} = G_{13} = 0.5E_2, \quad \nu_{12} = 0.25.$$

It is noted that the stiffness constants are calculated using the closed-form expressions presented in [36], which are based on classical lamination theory. Numerical results from the exact solution of Khdeir and Reddy [2] based on Euler–Bernoulli theory, trigonometric series analytical solution of Nguyen et al. [7] and finite element method (FEM) results of Murthy et al. [37] are shown in Tables 3 and 4 for symmetric and anti-symmetric beams respectively, and compared against the results from the current approach. It is observed that the results from the current approach are in excellent agreement with others. It is noteworthy that the exact solutions proposed in the current work simplify to exact solutions of Khdeir and Reddy [2] in the case of Euler–Bernoulli theory when twist φ and out-of-plane displacement v and associated stiffness and coupling terms are absent. However, slight differences (generally less than 0.15%) are observed between the results based on the exact solution of the current approach and those based on the exact solution of Khdeir and Reddy [2] which can be attributed to different values used for the bending stiffness and/or bending–axial coupling of beams.

Table 3
Normalised mid-span displacements $\overline{w}$ of [0°/90°/0°] composite beam ($\ell/h = 50$).

	$H-H$	$C-H$	$C-C$	$C-F$
Exact [2]	0.646	0.259	0.129	2.198
Analytical [7]	0.665	–	0.147	2.250
FEM [37]	0.661	0.279	0.147	2.262
CCM (Present)	0.647	0.259	0.129	2.201
Closed-Form (Present)	0.647	0.259	0.129	2.201

Table 4
Normalised mid-span displacements $\overline{w}$ of [0°/90°] composite beam ($\ell/h = 50$).

	$H-H$	$C-H$	$C-C$	$C-F$
Exact [2]	3.322	1.329	0.664	11.293
Analytical [7]	3.336	–	0.679	11.335
FEM [37]	3.318	1.343	0.681	11.392
CCM (Present)	3.325	1.330	0.665	11.305
Closed-Form (Present)	3.325	1.330	0.665	11.305

7. Benchmark problems

In this section examples of deflection of anisotropic composite beams subject to different boundary conditions under the action of distributed load are presented.

A 1 m long laminated fibre-reinforced slender beam with a 100 mm× 0.75 mm rectangular cross-section is considered for all numerical samples. The material and geometric properties used in the samples are as follows (material properties are reported from [38] for HexPly HTA 6376):

$$E_{11} = 135.64\ \text{GPa}, E_{22} = 10.14\ \text{GPa}, G_{12} = 5.86\ \text{GPa}, \nu_{12} = 0.29.$$

In order to study the effects of different coupling terms on the deflection of a composite beam, the following stacking sequences are considered:

1. Symmetric $[45_3]_s$ with bending–twist coupling,
2. Antisymmetric $[45_3/-45_3]$ with axial–twist coupling,
3. Cross-ply $[0_3/90_3]$ with axial–bending coupling,
4. Unsymmetric $[60_3/30_3]$ with bending–twist, axial–twist and axial–bending coupling.

The stiffness matrices for each stacking sequence, obtained by using closed-form expressions in [36], are presented in Table 5.

It is noted that for the case of an antisymmetric beam, a distributed torque load q_φ is applied and normalised deflections are $\overline{u} = \dfrac{ubhE_{22}}{q_\varphi\ell}$, $\overline{\varphi} = \dfrac{\varphi bh^3G_{12}}{q_\varphi\ell^2}$ while for other stacking sequences q_z is applied and normalised deflections are defined as: $\overline{u} = \dfrac{ubhE_{22}}{q_z\ell^2}$, $\overline{\varphi} = \dfrac{\varphi bh^3G_{12}}{q_z\ell^3}$ and

Table 5
Stiffness matrices for different stacking sequences.

Stacking sequence	***A***	***B***	***D***
$[45_3]_s$	$\begin{bmatrix} 1101466 & 0 \\ 0 & 0.1764 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ -0.0591 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.0714 & 0 \\ 0 & 917.8885 \end{bmatrix}$
$[45_3/-45_3]$	$\begin{bmatrix} 1523947 & -236.4506 \\ -236.4506 & 0.2561 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.0612 & 0 \\ 0 & 1269.9560 \end{bmatrix}$
$[0_3/90_3]$	$\begin{bmatrix} 5481178 & 0 \\ 884.7508 & 0 \end{bmatrix}$	$\begin{bmatrix} 884.7508 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.2569 & 0 \\ 0 & 4567.6480 \end{bmatrix}$
$[60_3/30_3]$	$\begin{bmatrix} 1637714 & 157.9162 \\ 157.9162 & 0.1667 \end{bmatrix}$	$\begin{bmatrix} -160.2034 & 0 \\ -0.0721 & 0 \end{bmatrix}$	$\begin{bmatrix} 0.1009 & 0 \\ 0 & 1364.7620 \end{bmatrix}$

Table 6
Normalised tip deflections of a cantilever under uniformly distributed load.

		$\overline{u}$	$\overline{\varphi}$	$\overline{w}$
$[45_3]_s$	Closed-Form	0	−267.3533	103.5665
	CCM	0	−267.3533	103.5665
$[45_3/-45_3]$	Closed-Form	268.9276	563.4393	0
	CCM	268.9276	563.4393	0
$[0_3/90_3]$	Closed-Form	179.2851	0	46.85769
	CCM	179.2851	0	46.85769
$[60_3/30_3]$	Closed-Form	−122.0870	−240.6619	83.43932
	CCM	−122.0870	−240.6619	83.43932

Table 7
Normalised maximum deflection of composite beam simply supported at both ends under uniformly distributed load.

		$\overline{u}$	$\overline{\varphi}$	$\overline{w}$
$[45_3]_s$	Closed-Form	0	12.86304	8.395549
	CCM	0	12.86304	8.395549
$[45_3/-45_3]$	Closed-Form	67.23191	140.8598	0
	CCM	67.23191	140.8598	0
$[0_3/90_3]$	Closed-Form	8.625858	0	2.710546
	CCM	8.625858	0	2.710546
$[60_3/30_3]$	Closed-Form	5.873916	11.57885	6.151060
	CCM	5.873916	11.57885	6.151060

$\overline{w} = \dfrac{100\,wbh^3 E_{22}}{q_z \ell^4}$. In the following subsections the numerical results are presented for the selected benchmark problems.

7.1. Cantilever composite beam under distributed load

A cantilever composite beam subject to uniformly distributed load is considered here. Table 6 shows the values of tip deformation for the case of four different stacking sequences. The results obtained from the exact solution and the CCM match excellently. Figs. 2 through 4 depict the obtained results for the non-zero deflections of composite beam for different stacking sequences.

7.2. Simply supported composite beam under distributed load

A simply supported composite beam subject to uniformly distributed load is considered in this section. The current closed-form solution and the CCM were used to calculate the maximum deformation of a beam. Corresponding values are given in Table 7. Figs. 5 through 7 show non-zero deformations of the composite beam for different stacking sequences obtained by means of two methods. Again, in all cases excellent agreement is observed between the results obtained from the exact solution and CCM. For this problem, the loading and boundary conditions are symmetric. In the case of symmetric, cross-ply and unsymmetric stacking sequences, the maximum deformation for w occurs at the mid-span of the beam (i.e. at $x = \ell/2$). However, due to anisotropic effects, the maximum deformations in u and φ do not occur at the same location. This behaviour can be justified considering the

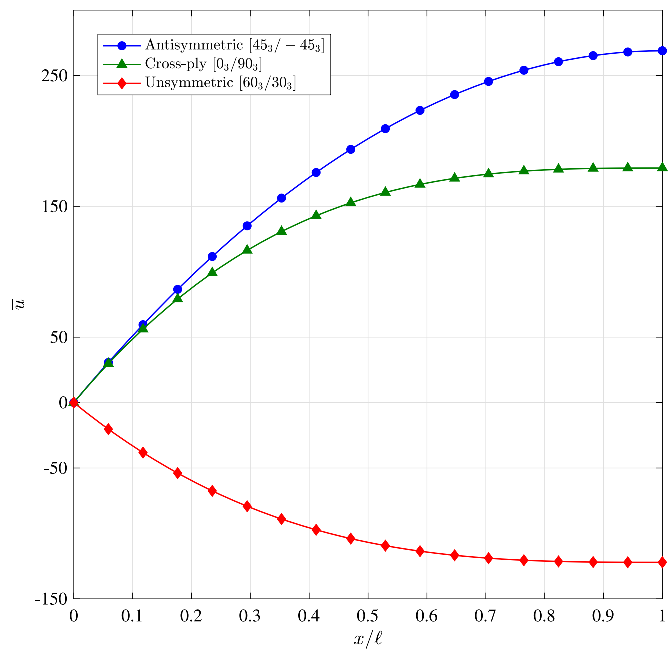

Fig. 2. Axial elongation of a cantilever under the action of uniformly distributed load for different stacking sequences.

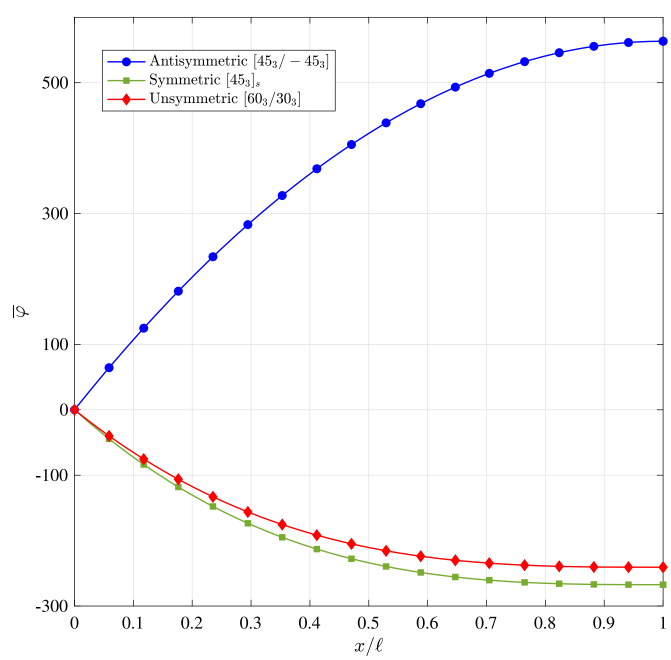

Fig. 3. Twist of a cantilever under the action of uniformly distributed load for different stacking sequences.

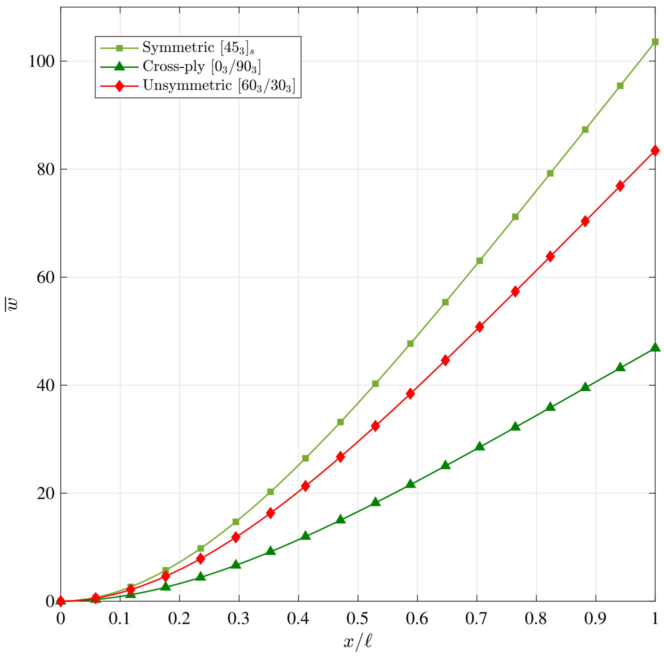

Fig. 4. Bending of a cantilever under the action of uniformly distributed load for different stacking sequences.

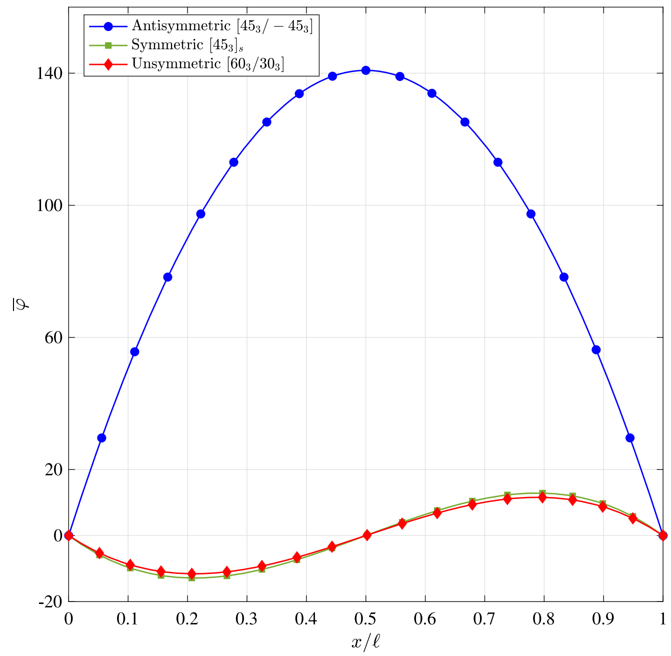

Fig. 6. Twist of a simply supported beam under the action of uniformly distributed load for different stacking sequences.

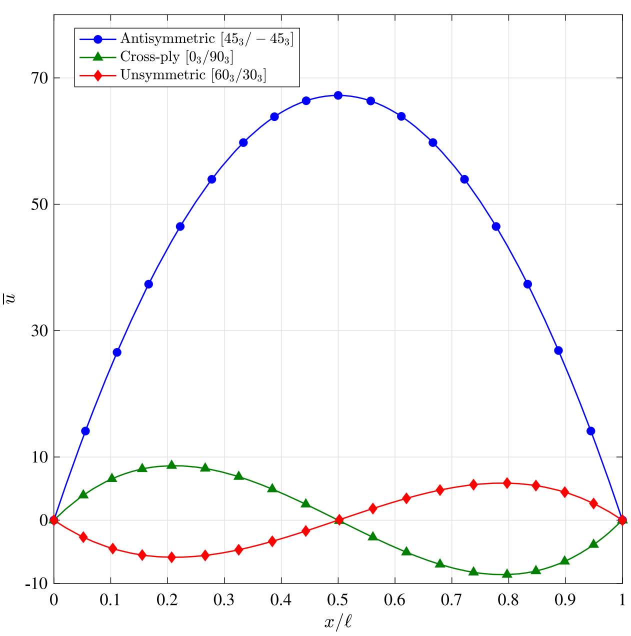

Fig. 5. Axial elongation of a simply supported beam under the action of uniformly distributed load for different stacking sequences.

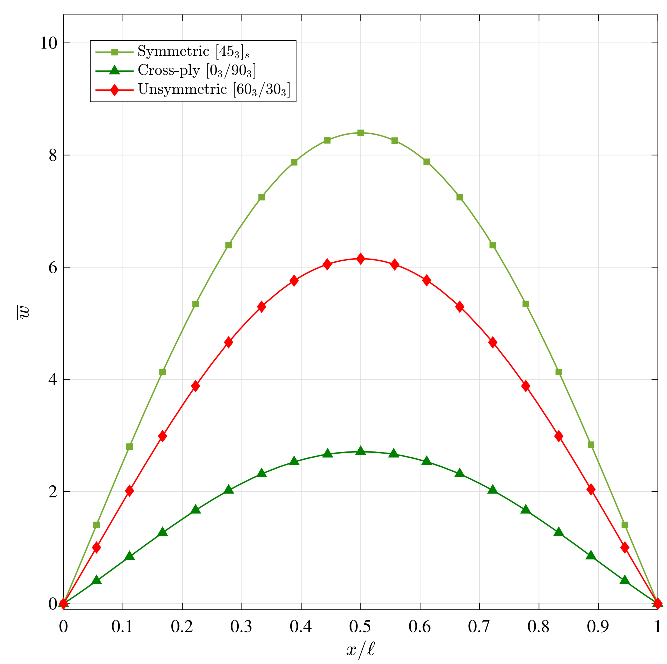

Fig. 7. Bending of a simply supported beam under the action of uniformly distributed load for different stacking sequences.

following information. Since the load is applied in the z direction, the deformations in u and φ occur as a result of the coupling terms. Due to the symmetric boundary conditions and the uniformly distributed applied load, the internal shear force in the z direction is distributed linearly along the beam span with a zero value at the mid-span. Thus, it is clear that w''' is also zero at the mid-span of the beam. From the constitutive equations, w''' is observed to be proportional to φ'' and/or u''. Consequently, at the mid-span the values of φ'' and/or u'' are zero, so a change of curvature in u and φ is expected at this point. Considering the symmetric boundary conditions and load, inevitably the deformation values of u and φ should be zero at the inflection point. The deformation in w occurs directly due to the applied symmetric distributed load, therefore this deformation is symmetric in the z direction with the maximum value at the middle of the beam. However, due to the fact that u and φ and their associated derivatives are linearly dependent, maximum deformations for these degrees of freedom in the case of the antisymmetric stacking sequence occur at the middle of the composite beam.

It is worth mentioning that the numerical examples presented in this section not only can be used for future benchmarking and validation purposes, but can also simulate various beam-like real-life engineering composite structures such as helicopter/wind turbine rotor blades and aircraft wings under the action of aerodynamic loads.

8. Conclusions

The static analysis of fully anisotropic Euler–Bernoulli composite beams under the action of uniformly distributed loads has been presented. Governing equations and boundary conditions are derived using the principle of virtual work and two different approaches are employed to obtain new closed-form analytical solutions. The first approach is based on the direct integration of the governing equations. In order to apply the direct integration technique, the governing equations are expressed in compact matrix form and using a novel eccentricity matrix the bending in the two principal directions are decoupled from in-plane and twist. Expressions for the constants of integration are provided for clamped–free, clamped–clamped, clamped–simply supported and simply supported–simply supported boundary conditions. The second approach is based on extracting the solutions from the previously obtained Timoshenko composite beam solutions by ignoring transverse shear effects and it is proven that they are identical to those obtained from the direct integration approach. Additionally, the Chebyshev collocation method is applied to solve the problem as an alternative numerical solution. Composite beams with various stacking sequences introducing different coupling terms were considered and the results were compared against other analytical as well as numerical results. Results for different boundary conditions show excellent agreement between closed-from and numerical solutions offering the prospect that the former can used to validate future numerical model development.

CRediT authorship contribution statement

Pedram Khaneh Masjedi: Conceptualization, Methodology, Software, Supervision, Writing - original draft, Writing - review & editing. **Olga Doeva:** Methodology, Software, Writing - original draft, Writing - review & editing. **Paul M. Weaver:** Supervision, Writing - review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Proof of Proposition 4.1

Let $\mathcal{C} = \mathcal{A}\mathcal{B}$ and $\mathcal{D} = \mathcal{A}^+\mathcal{B}^+$, then $\mathcal{C}^+ = (\mathcal{A}\mathcal{B})^+$. We need to prove that $\mathcal{C}^+ = \mathcal{D}$, i.e. $(\mathcal{A}\mathcal{B})^+ = \mathcal{A}^+\mathcal{B}^+$.

Using properties, specified in definition (4.13), and index notation for matrix multiplication:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj},$$

for any $i \neq n$ the following can be written:

$$\begin{aligned} d_{in} &= \sum_{k=1}^{n} a^+_{ik} b^+_{kn} \\ &= \sum_{k=1}^{n-1} a^+_{ik} b^+_{kn} + a^+_{in} b^+_{nn} \\ &= \sum_{k=1}^{n-1} a_{ik}(-b_{kn}) + (-a_{in}) b_{nn} \\ &= -(\sum_{k=1}^{n-1} a_{ik} b_{kn} + a_{in} b_{nn}) \\ &= -\sum_{k=1}^{n} a_{ik} b_{kn} \\ &= -c_{in} \\ &= c^+_{in}. \end{aligned}$$

For any $j \neq n$:

$$\begin{aligned} d_{nj} &= \sum_{k=1}^{n} a^+_{nk} b^+_{kj} \\ &= \sum_{k=1}^{n-1} a^+_{nk} b^+_{kj} + a^+_{nn} b^+_{nj} \\ &= \sum_{k=1}^{n-1} (-a_{nk}) b_{kj} + a_{nn}(-b_{nj}) \\ &= -(\sum_{k=1}^{n-1} a_{nk} b_{kj} + a_{nn} b_{nj}) \\ &= -\sum_{k=1}^{n} a_{nk} b_{kj} \\ &= -c_{nj} \\ &= c^+_{nj}. \end{aligned} \tag{A.1}$$

For any $p \neq n$, $q \neq n$:

$$\begin{aligned} d_{pq} &= \sum_{k=1}^{n} a^+_{pk} b^+_{kq} \\ &= \sum_{k=1}^{n-1} a^+_{pk} b^+_{kq} + a^+_{pn} b^+_{nq} \\ &= \sum_{k=1}^{n-1} a_{pk} b_{kq} + (-a_{pn})(-b_{nq}) \\ &= \sum_{k=1}^{n-1} a_{pk} b_{kq} + a_{pn} b_{nq} \\ &= \sum_{k=1}^{n} a_{pk} b_{kq} \\ &= c_{pq} \\ &= c^+_{pq}. \end{aligned} \tag{A.2}$$

For $i = j = n$:

$$\begin{aligned} d_{nn} &= \sum_{k=1}^{n} a^+_{nk} b^+_{kn} \\ &= \sum_{k=1}^{n} (-a_{nk})(-b_{kn}) \\ &= \sum_{k=1}^{n} a_{nk} b_{kn} \\ &= c_{nn} \\ &= c^+_{nn}. \end{aligned} \tag{A.3}$$

This implies that $\mathcal{C}^+ = \mathcal{D}$ or $(\mathcal{A}\mathcal{B})^+ = \mathcal{A}^+\mathcal{B}^+$. □

Appendix B. Stiffness properties of a composite strip [36]

$$EA = b\left(\overline{A}_{11} - \frac{\overline{B}_{12}^2}{\overline{D}_{22}}\right), \qquad GJ = 4b\left(\overline{D}_{66} - \frac{\overline{D}_{26}^2}{\overline{D}_{22}}\right),$$

$$EI_y = b\left(\overline{D}_{11} - \frac{\overline{D}_{12}^2}{\overline{D}_{22}}\right), \qquad EI_z = \frac{b^2}{12}EA$$

$$S_{ET} = -2b\left(\overline{B}_{16} - \frac{\overline{B}_{12}\overline{D}_{26}}{\overline{D}_{22}}\right), \qquad S_{EF} = b\left(\overline{B}_{11} - \frac{\overline{B}_{12}\overline{D}_{12}}{\overline{D}_{22}}\right),$$

$$S_{FT} = -2b\left(\overline{D}_{16} - \frac{\overline{D}_{12}\overline{D}_{26}}{\overline{D}_{22}}\right)$$

$$S_{EL} = S_{LT} = S_{FL} = 0$$

$$\overline{A}_{11} = A_{11} + \frac{A_{16}^2 A_{22} - 2A_{12}A_{16}A_{26} + A_{12}^2 A_{66}}{A_{26}^2 - A_{22}A_{66}}$$

$$\overline{B}_{11} = B_{11} + \frac{A_{12}A_{66}B_{12} + A_{16}A_{22}B_{16} - A_{26}\left(A_{16}B_{12} + A_{12}B_{16}\right)}{A_{26}^2 - A_{22}A_{66}}$$

$$\overline{B}_{12} = B_{12} + \frac{A_{12}A_{66}B_{22} + A_{16}A_{22}B_{26} - A_{26}\left(A_{16}B_{22} + A_{12}B_{26}\right)}{A_{26}^2 - A_{22}A_{66}}$$

$$\overline{B}_{16} = B_{16} + \frac{A_{12}A_{66}B_{26} + A_{16}A_{22}B_{66} - A_{26}\left(A_{16}B_{26} + A_{12}B_{66}\right)}{A_{26}^2 - A_{22}A_{66}}$$

$$\overline{D}_{11} = D_{11} + \frac{A_{66}B_{12}^2 - 2A_{26}B_{12}B_{16} + A_{22}B_{16}^2}{A_{26}^2 - A_{22}A_{66}}$$

$$\overline{D}_{12} = D_{12} + \frac{A_{66}B_{12}B_{22} + A_{22}B_{16}B_{26} - A_{26}\left(B_{16}B_{22} + B_{12}B_{26}\right)}{A_{26}^2 - A_{22}A_{66}}$$

$$\overline{D}_{22} = D_{22} + \frac{A_{66}B_{22}^2 - 2A_{26}B_{22}B_{26} + A_{22}B_{26}^2}{A_{26}^2 - A_{22}A_{66}}$$

$$\overline{D}_{16} = D_{16} + \frac{A_{66}B_{12}B_{26} + A_{22}B_{16}B_{66} - A_{26}\left(B_{16}B_{26} + B_{12}B_{66}\right)}{A_{26}^2 - A_{22}A_{66}}$$

$$\overline{D}_{26} = D_{26} + \frac{A_{66}B_{22}B_{26} + A_{22}B_{26}B_{66} - A_{26}\left(B_{26}^2 + B_{22}B_{66}\right)}{A_{26}^2 - A_{22}A_{66}}$$

$$\overline{D}_{66} = D_{66} + \frac{A_{66}B_{26}^2 - 2A_{26}B_{26}B_{66} + A_{22}B_{66}^2}{A_{26}^2 - A_{22}A_{66}}$$

where b is the width of the strip and A_{ij}, B_{ij} and D_{ij}, $i, j = 1, 2, 6$ are the entries of the well-known A, B and D matrices of classical lamination theory.

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